JUN 2 1 1928

PERIODICAL ROOM
GENERAL LIBRARY
UNIV. OF MICH.

AMERICAN
JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

EDITED BY

E. W. CHITTENDEN
UNIVERSITY OF IOWA

A. B. COBLE
THE JOHNS HOPKINS UNIVERSITY

ABRAHAM COHEN
THE JOHNS HOPKINS UNIVERSITY

G. C. EVANS
RICE INSTITUTE

FRANK MORLEY, CHAIRMAN
THE JOHNS HOPKINS UNIVERSITY

WITH THE COÖPERATION OF

E. T. BELL

F. D. MURNAGHAN

W. A. MANNING

PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

Volume L, Number 3

JULY, 1928

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

CONTENTS

On Hyperelliptic θ-Functions with Rational Characteristics. By OSCAR ZARISKI, ... 315

Certain Perfect Groups Generated by Two Operators of Orders Two and Three. By H. R. BRAHANA, ... 345

On Triadic Cremona Nets of Plane Curves. By FAY FARNUM, . . 357

Number Relations between Types of Extremals Joining a Pair of Points. By D. E. RICHMOND, ... 371

An Intrinsic Treatment of Poisson's Integral. By FRED W. PERKINS, 389

On the Invariant Combinants of Two Binary Quintics. By THOMAS W. MOORE, ... 415

A Boundary Value Problem of Ordinary Self-Adjoint Differential Equations with Singularities. By MARION CAMERON GRAY, . 431

Nets of Conics in the Real Domain. By ALAN D. CAMPBELL, . . 459

Rational Tacnodal and Oscnodal Quartic Curves Considered as Plane Sections of Quartic Surfaces. By L. T. MOORE and J. H. NEELLEY, ... 467

THE AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL for the current volume is $6.00 (foreign postage 25 cents); single numbers $1.50.

A few complete sets of the JOURNAL remain on sale.

Papers intended for publication in the JOURNAL may be sent to any of the Editors.

Editorial communications may be sent to Dr. A. COHEN at the Johns Hopkins University.

Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE, MARYLAND, U. S. A.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918

PRINTED BY THE J. H. FURST COMPANY
BALTIMORE, MD.

On Hyperelliptic θ-Functions with Rational Characteristics.

By Oscar Zariski.

1. *Preliminaries.* A θ-function of the arguments $v_1, \ldots, v_p$, attached to a definite normalized period matrix

$$|| 0, \cdots, 0, \pi i, 0, \cdots, 0; a_{j1}, \cdots, a_{jp} ||; \quad a_{ij} = a_{ji},$$

and of characteristic $[g_1, \ldots, g_p; h_1, \ldots, h_p]$, where the g's and the h's are real numbers, is designated by the symbol $\theta[g;h]((v_\sigma))$, and is defined by the following series:

$$\theta[g;h]((v_\sigma)) = \sum_{-\infty}^{\infty}{}_{(m)} \exp\Big[\sum_{1}^{p}{}_{i,j} a_{ij}(m_i+g_i)(m_j+g_j) + 2\sum_{1}^{p}{}_{i}(m_i+g_i)(v_i+h_i\pi i)\Big],$$

where in the first summation each of the p letters m_i runs through all integral values from $-\infty$ to $+\infty$. If ρ_{ij} denotes the real part of a_{ij}, the series is convergent for all finite values of the v's, as soon as $\sum_{1}^{p}{}_{i,j}\rho_{ij}x_ix_j$ is a definite negative quadratic form. If the elements g and h of the characteristic are rational numbers with the common denominator r, $g_i = \epsilon_i/r$, $h_i = \epsilon_i'/r$, then we will also designate the characteristic $[g;h]$ and the function $\theta[g;h]((v_\sigma))$ by the symbols $[\epsilon]_r$, $\theta[\epsilon]_r((v_\sigma))$.

A system of simultaneous linear combinations of the periods will be denoted by $\{g;h\}$:

$$\{g;h\} = \sum_{1}^{p}{}_{i} g_ia_{1i} + h_1\pi i, \cdots, \sum_{1}^{p}{}_{i} g_ia_{pi} + h_p\pi i,$$

and the corresponding period characteristic will be designated by $(g;h)$. In the case of rational period characteristics we will also use respectively the notations $\{\epsilon\}_r, (\epsilon)_r$.*

The θ-function $\theta[0;0]((v_\sigma))$ of characteristic zero will be simply designated by $\theta((v_\sigma))$. If the a_{ij}'s are the periods of a set of p normal abelian

* We use here the notations of A. Krazer, *Lehrbuch der Thetafunktionen*, Leipzig, Teubner, 1903. This treatise will be referred to in the sequel as (Kr.).

integrals u_i of the first kind, attached to an algebraic curve C of genus p, the function $\theta[g;h]((u_\sigma))$ of the arguments $u_1, \ldots, u_p$ is a Riemannian θ-function. The case which interests us is the one in which C is an hyperelliptic curve:

$$(1) \qquad y^2 = f(x) = (x-k)(x-k_1) \ \ldots \ (x-k_{2p+1}).$$

We construct the two-sheeted Riemann surface R of C by joining the sheets along $p+1$ non-intersecting cuts: $kk_1; k_2k_3; \ldots; k_{2p-2}, k_{2p-1}; k_{2p}, k_{2p+1}$. We define on R a set of $2p$ retrosections b_i, a_i $(i=1, \ldots, p)$ in the usual manner, namely: let b_i be a circuit in the first (upper) sheet surrounding the cut $k_{2i-2}k_{2i-1}$ (we put $k_0 = k$), and let a_i be a circuit which crosses the only two cuts $k_{2i-2}k_{2i-1}, k_{2p}k_{2p+1}$. Let $u_1, \ldots, u_p$ be a set of p abelian integrals of the first kind, normal with respect to the chosen set of retrosections, the period of u_i relative to b_j being πi if $i=j$, and 0 if $i \neq j$. Let a_{ij} denote the period of u_i relative to the retrosection a_j. Then $a_{ij} = a_{ji}$. With these conditions the u's are determined to within an additive constant, which we fix by assigning the branch point k_{2p+1} as common lower limit of the integrals.

If we put all the arguments u_σ equal to zero, the result $\theta[g;h]((0))$ is a function of the periods only, hence a function of the moduli of the curve C. In the hyperelliptic case $\theta[g;h]((0))$ is a function of the branch points $k, k_1, \ldots, k_{2p+1}$. These functions were first considered by Riemann,* who pointed out that every abelian integral of the second kind can be expressed by means of the logarithmic derivative of the function $\theta((u))$ and by means of $\theta((0))$, which appears in the additive constant of the integral. This remark of Riemann was the point of departure of two papers by Thomae, dedicated to the determination of $\theta((0))$ as a function of the moduli, or rather as a function of the branch points of the given algebraic function.† The same argument is treated by Fuchs.‡

Thomae finds a system of equations on partial derivatives which satisfies $\theta((0))$. In the hyperelliptic case Thomae was able to integrate completely

---

* "Theorie der Abelschen Funktionen," *Crelle's Journal*, Vol. 54 (1857), p. 151.

† "Bestimmung von $d \log \theta(0, 0, \ldots, 0)$ durch die Klassenmoduln, *Crelle's Journal*, Vol. 66 (1866), pp. 92-96; "Beitrag zur Bestimmung von $\theta(0, 0, \ldots, 0)$ durch die Klassenmoduln algebraischer Functionen," *Crelle's Journal*, Vol. 71 (1870), pp. 201-222.

‡ "Ueber die Form der Argumente der Thetafunctionen und über die Bestimmung von $\theta(0, 0, \ldots, 0)$ als Function der Klassenmoduln," *Crelle's Journal*, Vol. 73 (1871), pp. 305-323; "Ueber die linearen Differentialgleichungen, welchen die Periodicitätsmoduln der Abelschen Integrale genügen, und über verschiedene Arten von Differentialgleichungen für $\theta(0, 0, \ldots, 0)$" *Ibid.*, pp. 324-339.

these equations (see the second of the cited papers) and gave thus a remarkable expression not only of $\theta((0))$, but of any $\theta[\epsilon]_2((0))$. While the problem of expressing explicitly any hyperelliptic $\theta[\epsilon]_2((0))$ was thus completely solved by Thomae, the same problem for any rational characteristic $[\epsilon]_r (r>2)$ presents extreme difficulties. In another paper * Thomae treated and solved this problem for the functions $\theta[\epsilon]_3((0))$ in the elliptic case.

Without trying to write explicitly the expression of $\theta[\epsilon]_r((0))$, we want to investigate in this paper the nature of these functions limiting the argument to the hyperelliptic case. Especially we want to connect these functions with certain invariants of the polynomial $f(x)$ in equation (1), the vanishing of which admits an immediate geometric and algebraic interpretation. We prefer to explain our argument with a concrete instructive example:

Given a quartic plane curve C_4 without double points, one can draw 12 tangents to the curve from a generic point of the plane. However, if 12 lines of a pencil are taken in an arbitrary way, and if we look for a curve C_4 which should touch these lines (we put 12 conditions, while the quartics depend upon 14 arbitrary constants), no *irreducible* quartic will satisfy the required conditions. All solutions of the problem are furnished by degenerate C_4's which contain a double component. Therefore the 12 tangents to a C_4 constitute a particular projective set Γ_{12}. Indeed, the projectively distinct C_4's are ∞^6, hence the projectively distinct Γ_{12}'s of tangents are ∞^8. We deduce that these Γ_{12}'s have to satisfy *one* condition. From an algebraic point of view the peculiarity of the set of 12 tangents to a C_4 depends upon the fact that the discriminant $D(x)$ of the quartic in y is of the form

$$D(x) = i^3 - \lambda j^2, \tag{2}$$

where i and j are polynomials of degree 4 and 6 respectively, and λ is a constant factor. Now, the right-hand member of (2) contains 11 arbitrary constants; hence a generic polynomial of degree 12 cannot be represented as a sum of a square and a cube.

An intrinsic geometric explanation of the impossibility of constructing an irreducible C_4 which should touch 12 arbitrary lines of a pencil, is the following: The lines of a pencil cut out on the C_4 a linear involution g_4^1, which is not complete, because it is contained in the g_4^2 cut out by the lines of the plane. On the other hand, if we assign in advance the 12 branch points

* "Darstellung des Quotienten zweier Thetafunctionen, deren Argumente um Drittel ganzer Periodicitätsmoduln unterscheiden, durch algebraische Functionen," *Mathematische Annalen*, Vol. 6 (1873), pp. 603-612.

of a 4-valued algebraic function of genus 3, the corresponding g_4^1 on the quartic curve will be complete in general; i. e. for an arbitrary position of the branch points. Hence this g_4^1 will not be cut out by the lines of a pencil, but by conics through 4 points on the curve.

We observe that the particular Γ_{12}'s considered above, are also furnished by the 12 cubics with a double point belonging to a pencil of cubics, and also by the 12 tangents drawn to an hyperelliptic C_5 with a triple point from a point P on the curve. Indeed, the consideration of the cubic surface represented on a doubly-covered plane with a branch quartic curve C_4 leads to the obvious conclusion that to the 12 tangents to the C_4 correspond, on the surface, 12 cubics with a double point belonging to a pencil of plane sections. To prove the second remark we suppose that the point P is at infinity on the y-axis, and that the triple point is at the origin $x = y = 0$. The equation of C_5 is of the fourth degree in y, and the discriminant $D(x)$, of degree 18, is still of the form $D(x) = i^3 - \lambda j^2$. It is easily seen that $x = 0$ is a 6-fold root of $D(x) = 0$, and a double and triple root of $i = 0$, $j = 0$ respectively. Putting $D(x) = x^6 D_1(x)$, $i = x^2 i_1$, $j = x^3 j_1$, we have $D_1(x) = i_1^3 - \lambda j_1^3$, where the 12 roots of $D_1(x)$ correspond to the 12 tangents parallel to the y-axis.

The algebraic condition that a polynomial $f(x)$ should be of the form $i^3 - \lambda j^2$ is expressed by the vanishing of a certain invariant, which, however, it would not be easy to write explicitly. But a transcendental relation can be obtained by the following considerations:

We consider the hyperelliptic curve C of genus 5

$$x_3^2 \phi(x_1, x_2) = \psi(x_1, x_2), \tag{3}$$

where $\phi(x_1, x_2)$, $\psi(x_1, x_2)$ are binary forms of degree 5 and 7 respectively, and $\phi\psi = x_2^{12} f(x_1/x_2)$. The curve C possesses a 5-fold $x_1 = x_2 = 0$, the principal tangents at this point being given by the equation $\phi(x_1, x_2) = 0$. Let us consider the linear system ∞^7 of curves

$$x_3 \phi(x_1, x_2) = u(x_1, x_2), \tag{4}$$

where $u(x_1, x_2)$ is an arbitrary binary form of degree 6. The curves of this system cut out on C a complete series g_{12}^7, which, it may be observed, is the 6-fold of the hyperelliptic series g_2^1 cut out on C by the lines through the point $x_1 = x_2 = 0$. Eliminating x between (3) and (4) we obtain:

$$\phi(x_1, x_2)\psi(x_1, x_2) - u^2(x_1, x_2) = 0.$$

If, then, there exists a form u such that the first member of this equation becomes a perfect cube, the corresponding curve of the system (4) will have 4

contacts of 2nd order with the curve C. In other words, there will be a group of the g_{12}^{7} composed of 4 triple points, and this set of 4 points will not be made up of two pairs belonging to the g_2^1. Conversely, if such a trisection of the g_{12}^{7} is possible, then the form $\phi(x_1, x_2)\psi(x_1, x_2)$, or the polynomial $f(x)$, can be put into the form of the sum of a square and a cube.

Let us consider now the normal abelian integrals $u_\sigma (\sigma = 1, 2, \ldots, 5)$ of the curve C. The sum of the values of u_σ is constant for all groups of the g_{12}^{7}. If, according to the convention made above, the integrals vanish at the branch point k_{11}, this constant sum is equal to zero. If the series g_{12}^{7} contains a group made up of 4 triple points $\eta_1, \eta_2, \eta_3, \eta_4$, we will have

$$3\{u_\sigma(\eta_1) + u_\sigma(\eta_2) + u_\sigma(\eta_3) + u_\sigma(\eta_4)\} \equiv 0 \pmod{a_{\sigma j}}.$$

Hence

$$u_\sigma(\eta_1) + u_\sigma(\eta_2) + u_\sigma(\eta_2) + u_\sigma(\eta_4) \equiv \{\epsilon\}_3. \tag{5}$$

By a well-known theorem of Riemann,* $\theta((u_\sigma(\eta_1) + u_\sigma(\eta_2) + \cdots + u_\sigma(\eta_{p-1}) + K_\sigma))$ vanishes for every position of the $p-1$ points η_i. Here the K_σ's are the Riemannian constants. In the hyperelliptic case, the retrosections a_i, b_i being chosen as before, we have †

$$K_\sigma \equiv \{1, 1, \cdots, 1; p, p-1, \cdots, 1\}_2 = \{1^p; p, p-1, \cdots, 1\} = \{\kappa\}_2.$$

From (5) we deduce then, for $p = 5$, $\theta((\{\epsilon\}_3 + \{\kappa\}_2)) = 0$.

Since $\theta[g; h]((u_\sigma))$ differs from $\theta((u_\sigma + \{g; h\}))$ by a non-vanishing factor only, we can write the last relation thus

$$\theta[\{\kappa\} + \{\epsilon\}_3]((0)) = 0, \tag{6}$$

where by $[\kappa + \{\epsilon\}_3]$ we denote the sum of the characteristics (κ), $(\epsilon)_3$. The relation (6) expresses in transcendental form the required condition. The left-hand member is a function of the branch points k_i. We obtain a symmetric rational function of the k_i's by taking the product of all functions $\theta[\kappa + \{\epsilon\}_3]((0))$ relative to the $3^{10} - 1$ different characteristics $[\epsilon]_3$. This product will contain the desired invariant of the polynomial $f(x)$ as a factor. Starting from these facts which will be developed in the sequel, we will be able to calculate the degree of this invariant.

2. *Interpretation of the relation $\theta[\kappa + \{\epsilon\}_r]((0)) = 0$ for any value of p and r.* Suppose that the moduli of the curve of equation (1) satisfy the relation

$$\theta[\kappa + \{\epsilon\}_r]((0)) = 0, \tag{7}$$

* See, for instance (Kr.), p. 427.

† (Kr.), p. 459.

where $\kappa = [\kappa]_2$ is the characteristic $[1^p;\ p,\ p-1,\ \cdots,\ 1]$, and $[\epsilon]_r$ is a rational characteristic. We deduce then that there exist $p-1$ points $\eta_1, \ldots, \eta_{p-1}$ on C such that

$$\sum_1^{p-1}{}_\nu\, u_\sigma(\eta_\nu) \equiv \sum_1^{p-1}{}_\mu\, (\epsilon_\mu/r)a_{\sigma\mu} + (\epsilon_\sigma'/r)\pi i, \quad (\sigma = 1,\ 2,\ \cdots,\ p).$$

Hence $r\sum_1^{p-1}{}_\nu\, u_\sigma(\eta_\nu) \equiv 0.$

We consider two cases: $r(p-1)$ is 1) even, 2) odd.

1) $r(p-1) = 2m$. In this case the complete series $|\, g_{2m}\,|$ determined by the set of the $p-1$ r-fold points η_i is the m-fold of the hyperelliptic series $g_2{}^1$. We transform C into a curve $\bar{C}$ of order $p+2$ with a p-fold point at the origin. The equation of $\bar{C}$ in homogeneous coordinates will be the following:

$$x_3{}^2\phi_p(x_1, x_2) = \psi_{p+2}(x_1, x_2), \tag{8}$$

where ϕ_p and ψ_{p+2} are binary forms of degree p and $p+2$ respectively, and

$$\phi_p(x_1, x_2)\,\psi_{p+2}(x_1, x_2) = x_2{}^{2p+2}\, f(x_1/x_2) = f(x_1, x_2).$$

The complete series mentioned is a g_{2m}^{2m-p} and is cut out on $\bar{C}$ by the curves of the system

$$x_3\phi_p(x_1, x_2)\phi_{m-p-1}(x_1, x_2) = \psi_m(x_1, x_2), \tag{9}$$

where ϕ_{m-p-1} and ψ_m are arbitrary binary forms of degree $m-p-1$, m respectively. Indeed, the system (9) contains degenerate curves $\psi_m = 0$, composed of m arbitrary lines through the origin $x_1 = x_2 = 0$. Eliminating x_3 between (8) and (9) we obtain an equation of degree $2m$:

$$\phi_p\psi_{p+2}\cdot\phi_{m-p-1} - \psi^2{}_m = 0.$$

Since the series contains a set made up of $p-1$ r-fold points, the left-hand member of the last equation becomes a perfect r-power, when ϕ_{m-p-1}, ψ_m are conveniently chosen. We have then the following result:

If relation (7) holds, and if $r(p-1) = 2m$, then it is possible to find 3 binary forms $\phi_{m-p-1}, \psi_m, \psi_{p-1}$, the degrees of which are indicated by the subscripts, such that

$$\phi^2{}_{m-p-1}\, f = \psi^2{}_m + \psi^r{}_{p-1}. \tag{10}$$

A direct computation of constants shows that (10) implies effectively *one* condition on the coefficients of f. If $r = 3$ and $p = 5$, we have the case of the 12 tangents to a C_4, considered above. An analogous geometric inter-

pretation can be given for any odd p and $r = 3$. Consider a curve L of order $(p+3)/2$ possessing a $(p-5)/2$-fold point P and $(p-5)/2$ double points Q_i. The genus of L is $p-2$. The lines through P cut out a g_4^1 on L, whence $2p+2$ tangents can be drawn to the curve through P. The set of these tangents satisfy one condition, given by (10). Indeed, we can suppose that the equation of L is of degree 4 in one of the variables, say y. The discriminant $D(x)$ is the sum of a square and a cube, and at the same time $D(x)$ is the product of the square of a polynomial of degree $m-p-1 = (p-5)/2$ (which corresponds to the $(p-5)/2$ lines joining P to the double points Q_i) and the polynomial of degree $2p+2$.

2) Suppose now that $r(p-1)$ is odd and put $r(p-1) = 2m+1$. In this case we obtain the complete series $|\, g_{2m+1} \,|$ determined by the $p-1$ r-fold points η_i, by adding to the m-fold of the g_2^1 the common lower limit k_{2p+1} of the integrals u_σ. We consider again the transformed curve $\bar{C}$ of equation (8), and we suppose that $\phi_p(x_1, x_2)$ contains the factor $x_1 - k_{2p+1}x_2$.

The complete series mentioned above is a g_{2m+1}^{2m+1-p} and is cut out on $\bar{C}$ by the curves of the system

$$(9') \qquad [x_3\phi_p(x_1, x_2)/(x_1 - k_{2p+1}x_2)] \cdot \phi_{m-p}(x_1, x_2) = \psi_m(x_1, x_2),$$

where ϕ_{m-p} and ψ_m are arbitrary binary forms of degree $m-p$ and m respectively. Eliminating x_3 between (8) and (9′) we obtain an equation of degree $2m+1$:

$$\phi^2{}_{m-p}\phi_p\psi_{p+2}/(x_1 - k_{2p+1}x_2) - (x_1 - k_{2p+1}x_2)\psi_m{}^2 = 0.$$

We deduce, as before, that it is possible to find 3 binary forms $\phi_{m-p}, \psi_m, \psi_{p-1}$ such that

$$\phi^2{}_{m-p}f/(x_1 - k_{2p+1}x_2) = (x_1 - k_{2p+1}x_2)\psi^2{}_m + \psi^r{}_{p-1}.$$

In this relation the roots of the polynomial f do not appear symmetrically. Hence we can draw the conclusion in advance that if $r(p-1)$ is an odd number, the $r^{2p}-1$ functions

$$\theta[\kappa + \{\epsilon\}_r]((0))$$

are permuted among themselves only by those substitutions of the branch points which leave fixed the point k_{2p+1}. This statement will be proved later in a direct way.

3. *The formulas of Thomae.* In view of later applications we develop here briefly the main result stated by Thomae in the second of his cited papers.

The values of the integrals u_σ at the branch points $k, k_1, \cdots, k_i, \cdots,$

k_{2p+1} constitute systems of half-periods $\{a_i\}_2$, the characteristics of which are indicated by the following table: *

$$(11)\quad \begin{cases} (a) = (1, 0^{p-1}; 0^p); & (a_1) = (1, 0^{p-1}; 1, 0^{p-1}); \\ (a_2) = (0, 1, 0^{p-2}; 1, 0^{p-1}); & (a_3) = (0, 1, 0^{p-2}; 1, 1, 0^{p-2}); \\ \vdots \quad \vdots & \vdots \quad \vdots \\ (a_{2\rho-2}) = (0^{\rho-1}, 1, 0^{p-\rho}; 1^{\rho-1}, 0^{p-\rho+1}); & (a_{2\rho-1}) = (0^{\rho-1}, 1, 0^{p-\rho}; 1^{\rho}, 0^{p-\rho}); \\ \cdot \quad \cdot & \cdot \quad \cdot \\ (a_{2p-2}) = (0^{p-1}, 1; 1^{p-1}, 0); & (a_{2p-1}) = (0^{p-1}, 1; 1^p); \\ (a_{2p}) = (0^p; 1^p); & (a_{2p+1}) = (0^p; 0^p). \end{cases}$$

Here, for instance, $(a_{2\rho-2}) = (0^{\rho-1}, 1, 0^{p-\rho}; 1^{\rho-1}, 0^{p-\rho+1})$ means that

$$u_\sigma(k_{2\rho-2}) \equiv a_{\sigma\rho}/2 + \mu_\sigma \pi i/2,$$

where $\mu_\sigma = 1$, if $\sigma < \rho$, and $\mu_\sigma = 0$, if $\sigma \geqq \rho$.

We point out that the $2p+1$ characteristics (a_i) $(i = 0, 1, \cdots, 2p)$ constitute a fundamental system and that the $(a_{2\nu})$ are even characteristics, while the $(a_{2\nu-1})$ are odd. Moreover we have

$$(11')\quad (a) + (a_2) + \ldots + (a_{2p}) = (a_1) + (a_3) + \ldots + (a_{2p-1}) = (1^p; p, p-1, \cdots, 1) = (\kappa).$$

Let us consider the following p integrals of the first kind

$$W_\sigma = \int (x^{\sigma-1}/y)\,dx,$$

and let

$$A_{\sigma\mu} = \int_{b_\mu} (x^{\sigma-1}/y)\,dx,$$

be the period of W_σ relative to the retrosection b_μ.

We denote by $D(k, k_2, \cdots, k_{2p})$ the product of all differences $(k_{2\rho} - k_{2\rho'})$, where $\rho < \rho'$, and likewise by $D(k_1, k_3, \cdots, k_{2p+1})$ the product of all differences $(k_{2\rho+1} - k_{2\rho'+1})$, where $\rho < \rho'$. Thomae proves the following formula:

$$(12)\quad \theta((0)) = (|A_{\sigma\mu}|/(2\pi i)^p)^{1/2} \times [D(k, k_2, \ldots, k_{2p}) \cdot D(k_1, k_3, \ldots, k_{2p+1})]^{1/4},$$

* See (Kr.), p. 448. In this table the denominator 2 is omitted.

where $|A_{\sigma\mu}|$ denotes the determinant of the periods $A_{\sigma\mu}$. Let the branch points be distributed in two sets

$$k_{i_1}, k_{i_2}, \cdots, k_{i_{p+1}};$$

$$k_{j_1}, k_{j_2}, \cdots, k_{j_{p+1}};$$

where $i_1 < i_2 < \ldots < i_{p+1}$; $j_1 < j_2 < \ldots < j_{p+1}$. We have

$$\sum_1^{p+1}{}_\nu u_\sigma(k_{i_\nu}) \equiv \sum_1^{p+1}{}_\nu u_\sigma(k_{j_\nu}) \equiv \sum_1^{p+1}{}_\nu \{a_{i_\nu}\} = \{\epsilon\}_2.$$

Thomae finds the following formula, analogous to (12):

$$(12') \qquad \theta[\epsilon]_2((0)) = (|A_{\sigma\mu}^{\cdot}|/(2\pi i)^p)^{\frac{1}{2}} \times [D(k_{i_1}, k_{i_2}, \cdots, k_{i_{p+1}}) \cdot D(k_{j_1}, k_{j_2}, \cdots, k_{j_{p+1}})]^{\frac{1}{4}}.$$

The formulas (12) and (12′) express as functions of the branch points all those functions $\theta[\epsilon]_2((0))$, which do not vanish identically. Their number is, as known, equal precisely to $\binom{2p+1}{p}$.

4. $\theta[\epsilon]_r((0))$ *as function of the branch points.* Let $\theta[\epsilon]_r((0))$ be any θ-function with a rational characteristic, and let us consider the ratio

$$\theta[\epsilon]_r((0))/\theta((0)).$$

We prove that it is an *algebraic* function of the branch points. Indeed, if $\delta, \delta_1, \delta_2, \ldots, \delta_p$ are $p+1$ arbitrary points of the curve C, and if we have at $\delta_i: x = x_i, y = y_i$, the ratio

$$\theta[\epsilon]_r((u_\sigma(\delta) - \sum_1^p{}_i u_\sigma(\delta_i) - K_\sigma))/\theta((u_\sigma(\delta) - \sum_1^p{}_i u_\sigma(\delta_i) - K_\sigma))$$

is the rth root of a rational function Q of each of the points (x, y); (x_i, y_i), which, as function of δ, becomes ∞ of order r at the p points $\delta_1, \ldots, \delta_p$, and 0 of order r at the p zero's $\eta_1, \ldots, \eta_p$ of the numerator. The construction of such a function requires algebraic operations only. Let a function $R(x, y; x_1, y_1; \ldots; x_p, y_p)$ which is to satisfy the same conditions as Q, be constructed algebraically. Then R will depend algebraically upon the branch points, and will differ from Q by a factor independent of x and y:

$$Q = B(x_1, y_1; \ldots; x_p, y_p) R(x, y; x_1, y_1; \ldots; x_p, y_p).$$

We have only to prove that B, which is a rational function of (x_i, y_i), depends

algebraically upon the branch points. This is shown by the following method of calculating B, indicated by Riemann.* We interchange, in the expression of Q, δ and δ_1, and substitute instead of $\delta_1, \ldots, \delta_p$ the point η_1 and the points $\bar{\eta}_2, \ldots, \bar{\eta}_p$, which are the conjugates of $\eta_2, \ldots, \eta_p$ in the hyperelliptic g_2^1. We have then, assuming that at $\eta_i: x = x_i', y = y_i'$:

$$Q(x_1', y_1'; x, y; x_2', -y_2'; \cdots; x_p', -y_p')$$

$$= \frac{\theta[\epsilon]_r((u_\sigma(\eta_1) - u_\sigma(\delta) - \sum_2^p{}_i\, u_\sigma(\bar{\eta}_i) - K_\sigma))}{\theta((u_\sigma(\eta_1) - u_\sigma(\delta) - \sum_2^p{}_i\, u_\sigma(\bar{\eta}_i) - K_\sigma))}.$$

Due to the relations

$$\sum_2^p{}_\nu\, u_\sigma(\bar{\eta}_\nu) + K_\sigma \equiv -\sum_2^p{}_\nu\, u_\sigma(\eta_\nu) - K_\sigma,$$

$$\sum_1^p{}_\nu\, u_\sigma(\eta_\nu) \equiv \sum_1^p{}_\nu\, u_\sigma(\delta_\nu) - \sum_1^p{}_\mu(\epsilon_\mu/r)a_{\mu\sigma} - (\epsilon_\sigma'/r)\pi i,$$

the last ratio can be transformed, and the result is that

$$Q(x_1', y_1'; x, y; x_2', -y_2'; \cdots; x_p', -y_p') = \omega/Q(x, y; x_1, y_1; \cdots; x_p, y_p),$$

where ω is a root of unity. Hence

$$\omega/B^2 = R(x, y; x_1, y_1; \cdots; x_p, y_p)\ R(x_1', y_1'; x, y; x_2', -y_2'; \cdots; x_p', -y_p'),$$

and therefore B depends algebraically upon the branch points.

Suppose now that the points δ, δ_i coincide with the branch points $k_{2p+1}, k_1, k_3, \ldots, k_{2p-1}$ respectively. We obtain then, by means of relation (11′) of the preceding paragraph,

$$\theta[\epsilon]_r((0))/\theta((0)) = \omega'[B(k_1, 0; k_3, 0; \cdots; k_{2p-1}, 0) \cdot R(k_{2p+2}, 0; k_1, 0; \cdots; k_{2p-1}, 0)]^{1/r},$$

where ω' is a root of unity, and this proves that the left-hand member is an algebraic function of the branch points.

Hence, from the formula of Thomae (12) we deduce that any $\theta[\epsilon]_r((0))$ *is the product of the transcendental factor* $|A_{\sigma\mu}|^{\frac{1}{2}}$ *and of an algebraic function of the branch points.*

The number of all functions $\theta[\epsilon]_r((0))$, if the characteristic is reduced

---

* See the cited paper, "Theorie der Abelschen Functionen," pp. 154-155.

modulo r, is equal to $r^{2p}-1$, if r is a prime number. If r is composite, and $q_1, q_2, \ldots, q_\nu$ are its distinct prime factors, then the said number, which we denote by N_r, is given by the formula

$$N_r = \phi(r) - \sum \phi(r/q_1) + \sum \phi(r/q_1q_2) - \cdots + (-1)^\nu \phi(r/q_1q_2\cdots q_\nu), \tag{13}$$

where $\phi(\alpha) = \alpha^{2p}-1$.

Suppose now that the set of the branch points varies continuously, starting from an initial position and returning to it, the result being a permutation (perhaps the identity) of the branch points. What is the effect on a given $\theta[\epsilon]_r((0))$ of such a cyclic variation of the branch points? Are the N_r functions $\theta[\epsilon]_r((0))$, or rather their algebraic factors, permuted among themselves? The answer is negative. A continuous cyclic variation of the set of the branch points defines a transformation of the retrosections a_i, b_i, and hence a transformation of the periods. Any $\theta[\epsilon]_r((0))$ is transformed, to within a transcendental factor *independent of the characteristic* (and therefore due to the presence of the factor $|A_{\sigma\mu}|^{\frac{1}{2}}$ in the expression of $\theta[\epsilon]_r((0))$), into a new function $\theta[\bar{\epsilon}]_{\bar{r}}((0))$. The characteristics are subjected to a linear *non-homogeneous* transformation: there is an additive term, the effect of which is the addition of a characteristic $[\eta]_2$.* Hence generally $\bar{r} = r$.

One may have the impression that in order to obtain a closed field of θ-functions, it is necessary to consider all functions $\theta[\{\eta\}_2 + \{\epsilon\}_r]((0))$, where $\{\eta\}_2$ runs through all 4^p characteristics $[\eta]_2$. However, a transformation of the periods defined by a continuous cyclic variation of the set of the branch points, is not the most general of its kind. As a matter of fact a closed field of θ-functions is obtained by considering a smaller number of them. It is convenient for our purpose to represent any characteristic in the form $[\kappa + \{\epsilon\}_r]$, where $[\kappa]$ is the fixed characteristic $[1^p; p, p-1, \ldots, 1]_2$.

The function $\theta[\kappa + \{\epsilon\}_r]((u_\sigma(\eta)))$ vanishes at p points $\eta_1, \ldots, \eta_p$ such that

$$\sum_1^p{}_i\, u_\sigma(\eta_i) \equiv -\sum_1^p{}_\nu\, (\epsilon_\nu/r)\cdot a_{\sigma\nu} - (\epsilon_\sigma'/r)\cdot \pi i.$$

Hence

$$r\sum_1^p{}_i\, u_\sigma(\eta_i) \equiv 0 \equiv rp\cdot u_\sigma(k_{2p+1}). \tag{14}$$

We consider first a cyclic variation of the set of the branch points, which leaves fixed the lower limit k_{2p+1} of the integrals u_σ. Then the transformed

* See (Kr.), chap. 5, especially p. 166, formula (XXVIII), and p. 181. Compare also Krazer and Prym, *Neue Grundlagen einer Theorie der allgemeinen Thetafunctionen*, Leipzig (1892), 2nd part, chap. 6.

integrals $\bar{u}_\sigma$ are homogeneous linear combinations of the old integrals u_σ. Let $\theta[\kappa + \{\bar{\epsilon}\}_{\bar{r}}]((u_\sigma(\eta)))_{(a)} = \theta[\kappa + \{\epsilon\}_r]((\bar{u}_\sigma(\eta)))_{(\bar{a})}$ be the transformed θ-function. Here the subscript $(\bar{a})$ indicates that the θ-function is defined by the new periods $\bar{a}_{ij}$. Then $\theta[\kappa + \{\epsilon\}_r]((0))_{(a)}$ is transformed into

$$\theta[\kappa + \{\bar{\epsilon}\}_{\bar{r}}]((0))_{(\bar{a})}.$$

The p zeros, say $\bar{\eta}_1, \ldots, \bar{\eta}_p$, of the transformed function

$$\theta[\kappa + \{\bar{\epsilon}\}_{\bar{r}}]((u_\sigma(\eta)))_{(a)}$$

satisfy the relations

$$\sum_1^p{}_i\, u_\sigma(\bar{\eta}_i) \equiv -\sum_1^p{}_\nu\, (\bar{\epsilon}_\nu/\bar{r})\, a_{\sigma\nu} - (\bar{\epsilon}_\sigma'/\bar{r})\pi i.$$

On the other hand, after the cyclic continuous variation of the branch points is completed, (14) becomes

$$r\sum_1^p{}_i\, u_\sigma(\bar{\eta}_i) \equiv rp \cdot u_\sigma(k_{2p+1}) \equiv 0.$$

Hence

$$r\,\{\bar{\epsilon}\}_{\bar{r}} \equiv \{0^p;\ 0^p\},$$

and we deduce $r = \bar{r}$.

Let us consider now a cyclic variation of the set of the branch points which carries k_{2p+1} into k_e. We have

$$u_\sigma(k_e) \equiv \sum_1^p{}_i\, (a_i/2)a_{\sigma i} + (a_\sigma'/2)\pi i,$$

where the integers a_i, a_i' are the elements of the characteristic (a_e) of the table (11). The transformed integrals $\bar{u}_\sigma$ are homogeneous linear combinations of the differences

$$u_\sigma - \sum_1^p{}_i\, (a_i/2)a_{\sigma i} - (a_\sigma'/2)\pi i.$$

The transformed function

$$\theta[\kappa + \{\epsilon\}_r]((\bar{u}_\sigma))_{(\bar{a})}$$

is, to within a factor, of the form

$$\theta[\kappa + \{\bar{\epsilon}\}_{\bar{r}}]\Big((u_\sigma - \sum_1^p{}_i\, (a_i/2)a_{\sigma i} - (a_\sigma'/2)\pi i)\Big), \tag{15}$$

and $\theta[\kappa + \{\epsilon\}_r]((0))$ is transformed into

$$\theta[\kappa + \{\bar{\epsilon}\}_{\bar{r}}]((0)).$$

The zero's $\bar{\eta}_i$ of the transformed function (15) satisfy the relations

$$\sum_1^p{}_i\, u_\sigma(\bar{\eta}_i) \equiv -\Big(\sum_1^p{}_\nu\, (\bar{\epsilon}_\nu/\bar{r})a_{\sigma\nu} + (\epsilon_\sigma'/\bar{r})\pi i\Big) + \Big(\sum_1^p{}_\nu\, (a_\nu/2)a_{\sigma\nu} + (a_\sigma'/2)\pi i\Big).$$

On the other hand we have by means of (14),

$$r\sum_1^p{}_i\, u_\sigma(\bar{\eta}_i) \equiv rp\cdot u_\sigma(k_e).$$

Hence we deduce

$$r\,\{\bar{\epsilon}\}_{\bar{r}} \equiv -r(p-1)\{a_e\}_2.$$

If $r(p-1)$ is even, we have $r\{\bar{\epsilon}\}_{\bar{r}} \equiv \{0^p; 0^p\}$, and hence $\bar{r} = r$.

If $r(p-1)$ is odd, we have $r\{\bar{\epsilon}\}_{\bar{r}} \equiv \{a\}_2$, hence $\{\bar{\epsilon}\}_{\bar{r}}$ is of the form $\{\eta\}_r + \{a\}_2$.

We have thus the following result:

If $r(p-1)$ is even, the N_r functions $\theta[\kappa + \{\epsilon\}_r]((0))$ are permuted among themselves, to within factors, for any cyclic variation of the set of the branch points.

If $r(p-1)$ is odd, a closed field of θ-functions is given by the following $2p+2$ sets of functions:

$$(16)\quad \theta[\kappa + \{\epsilon\}_r]((0)),\ \theta[\kappa + a + \{\epsilon\}_r]((0)),\ \theta[\kappa + a_1 + \{\epsilon\}_r]((0)),\ \cdots,\ \theta[\kappa + a_{2p} + \{\epsilon\}_r]((0)).$$

Here $a, a_1, \cdots, a_{2p}$ are the characteristics indicated in the table (11). We foresaw this result above (§ 2) from algebraic considerations.

The number of the functions, contained in the sets (16), is $(2p+2)N_r$.

The number of distinct functions in the two cases is $N_r/2$, $(p+1)N_r$, respectively, since $\theta[g; h]((0)) = \theta[-g; -h]((0))$.

The functions $\theta[\kappa + a_i + \{\epsilon\}_r]((0))$ of each set are permuted among themselves by any permutation of the branch points, which leaves fix the branch point k_i. In particular, we deduce that in any transformation of the periods, defined by a permutation of the branch points, the linear transformation of the characteristics, reduced modulo 2,

1) leaves unaltered the characteristic $[\kappa]$, if p is odd;
2) permutes the $2p+2$ characteristics

$$[\kappa],\ [\kappa + a],\quad [\kappa + a_1],\ \cdots,\ [\kappa + a_{2p}],$$

among themselves, if p is even.

This is in accordance with the fact, that, if p is odd, $p = 2s - 1$, the function $\theta[\kappa]((v))$ and all its partial derivatives of order $1, 2, \cdots, s-1$ vanish for $v = 0$, while the partial derivatives of order $s-1$ of any of the functions $\theta[\kappa + a_i]((0))$ do not vanish for $v = 0$. However, if p is even, $p = 2s$, the highest partial derivatives, which vanish, are in both cases of the same order, namely of order $s-1$.

We put $N_r' = N_r$, if $r(p-1)$ is even, and $N_r' = (2p+2)N_r$, if $r(p-1)$ is odd, and we consider the product of all N_r' θ-functions considered above. We suppose that the characteristics are all reduced in such a way that their elements are numbers $\geqq 0$ and < 1. Denoting this product by $\Pi_r\theta$ we have

$$\Pi_r\theta = |A_{\sigma\mu}|^{N_r'/2} F_r(k, k_1, \cdots, k_{2p+1}), \tag{17}$$

where, due to the preceding considerations, F_r is an algebraic function of the k's, which is reproduced, to within a factor (a root of unity) by any permutation of the branch points. Hence, a convenient power of F_r is a rational symmetric function of the k's. We will prove later *that F_r itself is an integral symmetric function of the k's.* Denoting by I_r the invariant of the polynomial $f(x)$, the vanishing of which implies the vanishing of one of the θ-functions of the product, F_r will contain as factor a certain power of I_r. Moreover, F_r contains as factor a certain power of the discriminant D of the polynomial $f(x)$, because several θ-functions vanish also when two branch points come together. It is easy to calculate the degree of F_r and also to prove that F_r contains as factor only the square of I_r. It is more difficult to calculate the power of D which appears in F_r, in order to determine the degree of I_r. For this purpose we study in the next section the behavior of a generic function $\theta[g; h]((0))$, of any characteristic, when two branch points come together.

5. *The behavior of $\theta[g; h]((0))$ when two branch points come together.* Since F_r is a symmetric function of the branch points, it is sufficient to restrict our consideration to one pair of branch points k_i, k_j. We choose the pair k, k_1, and while we fix the position of $k_1, k_2, \cdots, k_{2p+1}$, we conceive $\theta[g; h]((0))$ as a function of k only, and make k approach k_1. The behavior of $\theta[g; h]((0))$ in the case of coincidence of any other pair k_i, k_j can be determined by considering the transformation of the periods implied by the permutation $(k\,k_i)(k_1 k_j)$.

We look first at the behavior of the periods a_{ij}, when $k \to k_1$. The curve C

$$y^2 = (x-k)(x-k_1)\cdots(x-k_{2p+1})$$

tends to the curve $\bar{C}$,

$$\bar{y}^2 = (x - k_1)^2(x - k_2) \cdots (x - k_{2p+1}),$$

of genus $p - 1$. The $2p - 2$ retrosections $b_2, \cdots, b_p; a_2, \cdots, a_p$ of the Riemann surface R of C remain the retrosections of the Riemann surface $\bar{R}$ of $\bar{C}$. We denote the two overlapping points of $\bar{R}$ at $x = k_1, \bar{y} = 0$ by δ_1, δ_2; δ_1 being on the upper sheet. Denoting by $\bar{u}_\sigma$ the abelian integrals of $\bar{C}$ of the first kind normal with respect to the retrosections b_i, a_i $(i = 2, \cdots, p)$ and which vanish at k_{2p+1}, it is seen at once that

$$\lim_{k \to k_1} u_\sigma = \bar{u}_\sigma \qquad (\sigma = 2, \cdots, p). \tag{18}$$

As far as u_1 is concerned, we have

$$\lim_{k \to k_1} u_1 = \tfrac{1}{2}\tilde{\omega}_{\delta_1, \delta_2}, \tag{18'}$$

where $\tilde{\omega}_{\delta_1, \delta_2}$ is the normal abelian integral of the third kind on $\bar{C}$, having δ_1, δ_2 as singular points. Inded we have

$$u_\sigma = \int \sum_1^p{}_i \, \alpha_i^{(\sigma)} x^{i-1} dx/y,$$

where $\alpha_i^{(\sigma)}$ is reciprocal of $A_{i\sigma}$ in the determinant $| A_{i\sigma} |$. If $\lim_{k \to k_1} \alpha_i^{(\sigma)} = \bar{\alpha}_i^{(\sigma)}$, then

$$\lim_{k \to k_1} u_\sigma = \int \sum_1^p{}_i \, \bar{\alpha}_i^{(\sigma)} \, x^{i-1} dx/\bar{y}.$$

It can be proved directly that, if $\sigma \geqq 2$, the polynomial $\sum_1^p{}_i \, \bar{\alpha}_i^{(\sigma)} \, x^{i-1}$ is divisible by $x - k_1$, and, hence, that the limit considered is an integral of the first kind. However it is sufficient to note that, since any integral u_σ $(\sigma \geqq 2)$ satisfies the relation $\int_{b_1} du_\sigma = 0$, the same relation holds for their limits, when b_1 is replaced by a small circle of center δ_1. Hence the integral limits cannot have logarithmic singularities at δ_1 or δ_2. Since, moreover, the period of $\lim_{k \to k_1} u_\sigma$ relative to the retrosection b_ν is evidently πi, if $\nu = \sigma$, and 0 if $\nu \neq \sigma$, (18) follows. As far as (18′) is concerned, we observe that $\int_{b_1} du_1 = \pi i$, and passing to the limit, we deduce that $\lim_{k \to k_1} u_1$ differs from $\omega_{\delta_1, \delta_2}$ only by the factor 1/2. Denoting by $\bar{a}_{\sigma r}$ the periods of $\bar{u}_\sigma$, we have then

$$\lim_{k \to k_1} a_{\sigma r} = \bar{a}_{\sigma r} \qquad (\sigma, r = 2, 3, \cdots, p) \tag{19}$$

By a well-known theorem on the periods of the normal integral of the third kind we deduce

$$(19') \qquad \lim_{k\to k_1} a_{1r} = \bar{u}_r(\delta_2) - \bar{u}_r(\delta_1) = 2\bar{u}_r(\delta_2) \qquad (r = 2, 3, \cdots, p).$$

As far as a_{11} is concerned, it assumes infinitely great values when $k \to k_1$ and it must be pointed out that the real part of a_{11} tends to $-\infty$.

It is easily seen that all periods $a_{\sigma r}$, except a_{11}, are monodromic functions of k in the neighborhood of k_1, and that a_{11} increases by $2\pi i$, when k turns around k_1. Indeed, when k turns around k_1, the retrosections are transformed as follows:

$$b_i' = b_i \ (i = 1, 2, \cdots, p); \quad a_1' = a_1 + 2b_1; \quad a_j' = a_j \ (j = 2, \cdots, p).$$

We have then the following developments in the neighborhood of k_1:

$$a_{\sigma r} = \bar{a}_{\sigma r} + \alpha_1^{(\sigma r)}(k - k_1) + \alpha_2^{(\sigma r)}(k - k_1)^2 + \cdots \qquad (\sigma, r = 2, \cdots, p)$$

$$a_{1r} = 2\bar{u}_r(\delta_2) + \alpha_1^{(1r)}(k - k_1) + \cdots \qquad (r = 2, \cdots, p)$$

$$a_{11} = \log(k - k_1) + \alpha_0^{(11)} + \alpha_1^{(11)}(k - k_1) + \cdots$$

It may be useful to calculate some more coefficients of these developments. We prove:

$$(20) \qquad \alpha_1^{(\sigma r)} = 0$$

$$(20') \qquad \alpha_2^{(\sigma r)} = \tfrac{1}{2}[d\bar{u}_\sigma/dz \cdot d\bar{u}_r/dz]_{z=\delta_2} \qquad \Big\} \ (\sigma, r = 2, \cdots, p)$$

$$(20'') \qquad \alpha_1^{(1r)} = [d\bar{u}_r/dz]_{z=\delta_2} \qquad (r = 2, \cdots, p)$$

Let us differentiate u_σ with respect to k. We have

$$\frac{du_\sigma}{dk} = \tfrac{1}{2}\int \sum_1^p{}_i \, \alpha_i^{(\sigma)} \, x^{i-1}dx/(x - k)y + \int \sum_1^p{}_i \, \frac{d\alpha_i^{(\sigma)}}{dk} \, x^{i-1}dx/y,$$

whence du_σ/dk is an integral of the second kind, having a pole of the first order at the branch point k. Moreover, du_σ/dk possesses zero differences along the retrosections a_r, and the differences $da_{\sigma r}/dk$ along the retrosections b_r $(r = 1, 2, \cdots, p)$.*

From this we draw the following conclusion, which will be the base of the following considerations, namely, that the *limit of* du_σ/dk $(\sigma = 1, 2, \cdots, p)$ *for* $k \to k_1$ *cannot possess logarithmic singularities at the points* δ_1, δ_2. The same conclusion holds for all derivatives of higher order.

---

* See Thomae, second of the cited papers, p. 211.

Put, for brevity

$$\sum_1^p \alpha_i^{(\sigma)} x^{i-1} = \phi_\sigma(x) \qquad \lim_{k\to k_1} (d^\lambda \phi_\sigma(x)/dk^\lambda) = \bar{\phi}_\sigma^{(\lambda)}(x)$$

$$\lim_{k\to k_1} \phi_\sigma(x) = \bar{\phi}_\sigma(x) \qquad \lim_{k\to k_1} (d^\lambda u_\sigma/dk^\lambda) = \bar{u}_\sigma^{(\lambda)}.$$

We have then

$$\bar{u}_\sigma^{(1)} = \tfrac{1}{2}\int \bar{\phi}_\sigma(x)\,dx/(x-k_1)\bar{y} + \int \bar{\phi}_\sigma^{(1)}(x)\,dx/\bar{y}. \tag{21}$$

We saw before that $\bar{\phi}_\sigma(x)$ is divisible by $x-k_1$, if $\sigma \geqq 2$. Hence, the points δ_1, δ_2 are not poles of $\bar{u}_\sigma^{(1)}$. Since, on the other hand, $\bar{u}_\sigma^{(1)}$ cannot possess logarithmic singularities, it follows that $\bar{u}_\sigma^{(1)}$ is everywhere regular on $\bar{R}$. Since $\bar{u}_\sigma^{(1)}$ possesses zero differences along the retrosections a_σ $(\sigma = 2, \cdots, p)$, it follows that it is identically zero, and this proves the relation (20).

Consider now $\bar{u}_1^{(1)}$, which is an integral of the second kind, having two poles δ_1, δ_2. Since

$$\int \bar{\phi}_1^{(1)}(x)\,dx/\bar{y} = \tfrac{1}{2}\bar{\omega}_{\delta_1,\delta_2},$$

we deduce that the residues of $\bar{u}_1^{(1)}$ at δ_1 and δ_2 are $\frac{1}{4}$, $-\frac{1}{4}$ respectively. Hence, denoting by $\bar{t}_\delta$ a normal integral of the second kind, relative to the pole δ, we have

$$\bar{u}_1^{(1)} = \tfrac{1}{4}\bar{t}_{\delta_1} - \tfrac{1}{4}\bar{t}_{\delta_2}.$$

Since the constant difference of $\bar{t}_\delta$ along b_σ is $-2(d\bar{u}_\sigma/dz)_{z=\delta}$, we deduce

$$\lim_{k\to k_1} (da_{1\sigma}/dk) = -\tfrac{1}{2}(d\bar{u}_\sigma/dz)_{z=\delta_1} + \tfrac{1}{2}(d\bar{u}_\sigma/dz)_{z=\delta_2} = (d\bar{u}_\sigma/dz)_{z=\delta_2},$$
$$(\sigma = 2, \cdots, p),$$

and this proves (20″).

Consider now $\bar{u}_\sigma^{(2)}$ for $\sigma \geqq 2$. We have

$$\bar{u}_\sigma^{(2)} = \tfrac{3}{4}\int \bar{\phi}_\sigma(x)\,dx/(x-k_1)^2\bar{y} + \int \bar{\phi}_\sigma^{(1)}(x)\,dx/(x-k_1)\bar{y} + \int \bar{\phi}_\sigma^{(2)}(x)\,dx/\bar{y}.$$

Taking into consideration that $\bar{u}_\sigma^{(1)}$, given by (21), is identically zero, we transform $\bar{u}_\sigma^{(2)}$ and obtain

$$\bar{u}_\sigma^{(2)} = \tfrac{1}{4}\int \bar{\phi}_\sigma(x)\,dx/(x-k_1)^2\bar{y} + \int \bar{\phi}_\sigma^{(2)}(x)\,dx/\bar{y}.$$

We know in advance that the logarithmic terms in these two integrals eliminate each another, and hence $\bar{u}_\sigma^{(2)}$ possesses two poles of the first order at δ_1 and δ_2.

Since.

$$\bar{u}_\sigma = \int \bar{\phi}_\sigma(x)\,dx/\bar{y},$$

we deduce that the residues of $\bar{u}_\sigma^{(2)}$ at δ_1 and δ_2 are

$$-\tfrac{1}{4}(d\bar{u}_\sigma/dz)_{z=\delta_1},\ -\tfrac{1}{4}(d\bar{u}_\sigma/dz)_{z=\delta_2},$$

respectively. Hence

$$\bar{u}_\sigma^{(2)} = -\tfrac{1}{4}(d\bar{u}_\sigma/dz)_{z=\delta_1}\cdot\bar{t}_{\delta_1} - \tfrac{1}{4}(d\bar{u}_\sigma/dz)_{z=\delta_2}\cdot\bar{t}_{\delta_2},$$

and consequently

$$\lim_{k\to k_1}(d^2a_{\sigma r}/dk^2) = \tfrac{1}{2}(d\bar{u}_\sigma/dz\cdot d\bar{u}_r/dz)_{z=\delta_1} + \tfrac{1}{2}(d\bar{u}_\sigma/dz\cdot d\bar{u}_r/dz)_{z=\delta_2}$$
$$= (d\bar{u}_\sigma/dz\cdot d\bar{u}_r/dz)_{z=\delta_2} \qquad (\sigma, r = 2, \cdots, p);$$

which proves (20′).

One can prove in the same way

$$\lim_{k\to k_1}(d^3a_{\sigma r}/dk^3) = 3/2\,[d/dz(d\bar{u}_\sigma/dz\cdot d\bar{u}_r/dz)]_{z=\delta_2},$$
$$(\sigma, r = 2, \cdots, p).$$

We observe, without insisting upon it, that while the calculated coefficients are expressed by means of the values which the integrals $\bar{u}_\sigma$ and their derivatives assume at the point δ_2, the coefficients of the higher order depend also upon the coefficients of the development of the integral of the second kind t_{δ_2} at the point δ_2.

We investigate now the behavior of any $\theta[g;h]((0))$ for $k\to k_1$.

In order not to interrupt the later exposition we prove two preliminary lemmas, referred to a curve C, hyperelliptic or not.

LEMMA I. *If C is of general moduli and if $\theta((mu_\sigma + e_\sigma))$, where m and e_σ are constants, is identically zero, then m is necessarily a rational number.*

Indeed, suppose that we have identically, i. e. for any point ξ on C,

$$\theta((mu_\sigma(\xi) + e_\sigma)) = 0. \tag{22}$$

Then we may write

$$mu_\sigma(\xi) + e_\sigma \equiv \sum_1^{p-1} u_\sigma(\eta_i) + K_\sigma,$$

where the K_σ's are the Riemannian constants. By considering a cyclic path for the point ξ, we deduce from (22)

$$\theta((mu_\sigma(\xi) + m\Omega_\sigma + e_\sigma)) = 0, \tag{22′}$$

where the Ω_σ's constitute an arbitrary system of simultaneous periods, and hence

$$(23) \qquad mu_\sigma(\xi) + m\Omega_\sigma + e_\sigma \equiv \sum_1^{p-1} u_\sigma(\eta_i') + K_\sigma.$$

Now, the infinite sets

$$(23') \qquad m\Omega_1, m\Omega_2, \ldots, m\Omega_p,$$

cannot be all distinct, modulo the periods. In fact, if they were, two systems of periods Ω, Ω' could be found, such that the numbers of the set

$$m\Omega_1 + \Omega_1', m\Omega_2 + \Omega_2', \ldots, m\Omega_p + \Omega_p'$$

would be infinitesimals, or else differ by infinitesimals from any set of numbers given in advance. Hence, by (22'), $\theta((v))$ would vanish identically, as function of the arguments v, which is impossible.

Suppose that between the different sets (23') there are q distinct sets, modulo the periods. Then (23), for different systems Ω, and for a given ξ, leads to only q distinct sets of $p-1$ points η_i'. Denoting these $q(p-1)$ points by $\eta_i^{(j)}$ $(i = 1, 2, \ldots, p-1; j = 1, 2, \ldots, q)$, we have

$$\sum_1^q \sum_1^{p-1} u_\sigma(\eta_i^{(j)}) \equiv qmu_\sigma(\xi) + c_\sigma,$$

where the c_σ's are constants. We define thus an algebraic correspondence on C, on which to a point ξ correspond $q(p-1)$ points η. If C is of general moduli, this correspondence cannot be singular, and hence qm is an integer, i. e. m is a rational number.

A slight modification of this proof becomes necessary, if it happens that for any ξ and Ω the equation (23) does not determine uniquely the set of $p-1$ points η_i'. In this case, if s is the dimension of the complete series $|g_{p-1}|$ the sets of which satisfy equation (23), we have only to fix s of the points η_i': the remaining $p-s-1$ points are thus uniquely determined. One of the indices of the correspondence defined above becomes then $q(p-s-1)$.

We observe that the only condition required in the proof is that C should not possess singular correspondences. Since the general hyperelliptic curve satisfies this condition,* the lemma is valid also in the hyperelliptic case.

LEMMA II. *If m is a rational number and if $\theta[\epsilon]_r((mu_\sigma))$ is identically zero, then m is necessarily an integer and $r = 2$.*

---

* See A. Lefschetz, "A Theorem on Correspondences on Algebraic Curves," *American Journal of Mathematics*, Vol. 50, No. 2 (April, 1928), pp. 159-166.

It is enough to prove this statement for hyperelliptic curves. We suppose that the integrals u_σ have a common lower limit. Then, if $\theta[\epsilon]_r((mu_\sigma))$ is identically zero, we have also

$$\theta[\epsilon]_r((0)) = 0;$$

and this relation implies, as we saw above, a condition on the branch points, if $r > 2$. Hence $r = 2$. Now, let $m = q_1/q$. Then we deduce, as in (22′), that

$$\theta[\epsilon]_2((\{\eta\}_q)) = 0,$$

or,

$$\theta[\{\epsilon\}_2 + \{\eta\}_q]((0)) = 0,$$

where $[\eta]_q$ is any arbitrary characteristic relative to the denominator q. Hence, reasoning as above, we deduce $q \leqq 2$. If $q = 2$, then $[\{\epsilon\}_2 + \{\eta\}_2]$ is substantially any characteristic $[\eta]_2$, and it is impossible that all 4^p functions $\theta[\eta]_2((0))$ should vanish. Hence $q = 1$, and m is an integer.

We consider now a rational characteristic $[g; h]$ and study the behavior of the function $\theta[g; h]((0))$ when $k \to k_1$. Since the real part of a_{11} tends to $-\infty$, it follows that every term

$$\exp\left[\sum_{1}^{p}{}_{i,j}\, a_{ij}(m_i + g_i)(m_j + g_j) + 2\sum_{1}^{p}{}_{i}\,(m_i + g_i)h_i\pi i\right]$$

tends to zero, unless $g_1 = m_1 = 0$. Hence, if $g_1 \neq 0$, we have

$$\lim_{k \to k_1} \theta[g; h]((0)) = 0.$$

But the functions $\theta[g; h]((0))$, where $g_1 \neq 0$, are not the only ones which vanish for $k \to k_1$. Indeed, if $g_1 = 0$, we will have, by means of (19),

$$\lim_{k \to k_1} \theta[g; h]((0)) = \bar{\theta}[g_2, \cdots, g_p;\ h_2, \cdots, h_p]((0)),$$

where $\bar{\theta}$ denotes functions θ relative to the limit curve $\bar{C}$ of genus $p - 1$. If $[g_2, \cdots, g_p;\ h_2, \cdots, h_p]$ is a conveniently chosen characteristic $[\epsilon]_2$ (for instance, an odd characteristic), the second term of the last relation vanishes. In any case, if $\theta[g; h]((0))$ possesses a zero at $k = k_1$, we wish to determine the order of this zero. We adopt, for brevity, the notation

$$\phi \leq \psi,$$

to indicate that for $k \to k_1$ the limit of the ratio ϕ/ψ is a finite number. If this limit is $\neq 0$, we will write

$$\phi \sim \psi.$$

We saw above that

$$e^{a_{11}} \sim k - k_1.$$

We arrange all terms in the expression of $\theta[g;h]((0))$ with respect to m_1, and put

$$\sum_{-\infty}^{+\infty}{}_{(m_2,\ldots,m_p)} \exp\Big[\sum_1^p{}_{ij} a_{ij}(m_i+g_i)(m_j+g_j) + 2\sum_1^p{}_i (m_i+g_i)h_i\pi i$$
$$= A_{m_1} \exp\left[a_{11}(m_1+g_1)^2 + 2(m_1+g_1)h_1\pi i\right].$$

We have then

$$\theta[g;h]((0)) = \sum_{-\infty}^{+\infty}{}_{m_1} A_{m_1} \exp\left[a_{11}(m_1+g_1)^2 + 2(m_1+g_1)h_1\pi i\right]. \tag{24}$$

Due to (19) and (19′) the limit of the general term of A_{m_1} for $k \to k_1$ is

$$\exp\Big[\sum_2^p{}_{i,j} \bar{a}_{ij}(m_i+g_i)(m_j+g_j) + 2\sum_2^p{}_i (m_i+g_i)\left[2(m_1+g_1)\bar{u}_i(\delta_2) + h_i\pi i\right].$$

Hence

$$\lim_{k\to k_1} A_{m_1} = \bar{\theta}[g_2,\cdots,g_p;\, h_2,\cdots,h_p]((2(m_1+g_1)\bar{u}_\sigma(\delta_2))). \tag{25}$$

The general term of (24) has, at $k = k_1$, a zero of order $(m_1+g_1)^2$ at least. We obtain the smallest value of $(m_1+g_1)^2$ by putting $m_1 = 0$, if $g_1 < \frac{1}{2}$, and $m_1 = -1$, if $g_1 > \frac{1}{2}$. If $g_1 = \frac{1}{2}$, this minimum corresponds to both the values $0, -1$ of m_1.

Let $g_1 < \frac{1}{2}$. Then, due to (25),

$$\lim_{k\to k_1} \exp\left[-a_{11}g_1^2\right] \cdot \theta[g;h]((0))$$
$$= \exp\left[2g_1h_1\pi i\right] \cdot \bar{\theta}[g_2,\cdots,g_p;\, h_2,\cdots,h_p]((2g_1\bar{u}_\sigma(\delta_2)));$$

hence

$$\theta[g;h]((0)) \sim (k-k_1)^{g_1^2}, \tag{26}$$

unless

$$\bar{\theta}[g_2,\cdots,g_p;\, h_2,\cdots,h_p]((2g_1\bar{u}_\sigma(\delta_2))) = 0. \tag{27}$$

Likewise, we have for $g_1 > \frac{1}{2}$:

$$\lim_{k\to k_1} \exp\left[-a_{11}(1-g_1)^2\right] \cdot \theta[g;h]((0))$$
$$= \exp\left[-2(1-g_1)h_1\pi i\right] \cdot \bar{\theta}[g_2,\cdots,g_p;\, h_2,\cdots,h_p]$$
$$((-2(1-g_1)\bar{u}_\sigma(\delta_2)));$$

hence

$$\theta[g;h]((0)) \sim (k-k_1)^{(1-g_1)^2}, \tag{26′}$$

unless

$$(27') \qquad \bar{\theta}[g_2, \cdots, g_p; h_2, \cdots, h_p]((-2(1-g_1)\bar{u}_\sigma(\delta_2))) = 0.$$

Finally, if $g_1 = \frac{1}{2}$, we have

$$\lim_{k \to k_1} \exp[-a_{11}/4] \cdot \theta[g; h]((0))$$
$$= \exp[h_1\pi i]\bar{\theta}[g_2, \cdots, g_p; h_2, \cdots, h_p]((\bar{u}_\sigma(\delta_2))) +$$
$$+ \exp[-h_1\pi i]\bar{\theta}[g_2, \cdots, g_p; h_2, \cdots, h_p]((-\bar{u}_\sigma(\delta)));$$

hence

$$(26'') \qquad \theta[g; h]((0)) \sim (k - k_1)^{\frac{1}{4}},$$

unless

$$(27'') \qquad \exp[h_1\pi i] \cdot \bar{\theta}[g_2, \cdots, g_p; h_2, \cdots, h_p]((\bar{u}_\sigma(\delta_2))) +$$
$$+ \exp[-h_1\pi i] \cdot \bar{\theta}[g_2, \cdots, g_p; h_2, \cdots, h_p]((-\bar{u}_\sigma(\delta_2))) = 0.$$

Since the branch points k_i are independent of each other, the equalities (27), (27′), (27″) have to be considered as identities, i. e. as valid for any point δ_2 on $\bar{R}$. Hence, from the Lemma II we deduce, that (27) and (27′) are possible only if $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ is a $[\epsilon]_2$ and $g_1 = \frac{1}{2}$ or 0.

We prove that (27″) implies also that $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ should be a $[\epsilon]_2$ and that $\theta[g_2, \cdots, g_p; h_2, \cdots, h_p]((\bar{u}_\sigma(\delta)))$ itself should vanish identically. In fact, suppose that $[g_2, \cdots, g_p; h_2, \cdots, h_p] = [\epsilon]_r$, and $r > 2$. The function $\bar{\theta}[g; h]((\bar{u}_\sigma(\delta)))$ cannot vanish identically, if the branch points of $\bar{C}$ are in arbitrary position (Lemma II). For the same reason $\bar{\theta}[g; h]((-\bar{u}_\sigma(\delta))) = \bar{\theta}[-g; -h]((\bar{u}_\sigma(\delta)))$ does not vanish identically. Each one of them possesses therefore $p-1$ zeros, say $\delta_1, \cdots, \delta_{p-1}$; $\delta_1', \cdots, \delta_{p-1}'$ respectively. We have

$$\sum_1^{p-1} \bar{u}_\sigma(\delta_i) \equiv -\sum_2^p g_j\bar{a}_{\sigma j} - h_\sigma\pi i + \bar{K}_\sigma,$$

$$\sum_1^{p-1} \bar{u}_\sigma(\delta_i') \equiv \sum_2^p g_j\bar{a}_{\sigma j} + h_\sigma\pi i + \bar{K}_\sigma,$$

where the $\bar{K}_\sigma$'s are the Riemannian constants, i. e. a system of half periods of characteristic $[1^{p-1}; p-1, p-2, \cdots, 1]$. If now (27″) holds, the zeros of both functions coincide, and therefore

$$2[\sum_2^p g_j\bar{a}_{\sigma j} + h_\sigma\pi i] \equiv 0, \qquad (\sigma = 2, 3, \cdots, p).$$

Hence $\{g_2, \cdots, g_p; h_2, \cdots, h_p\}$ is a half period, which contradicts our hypothesis. Hence $r = 2$.

We prove now that $\bar{\theta}[g;h]((\bar{u}_\sigma(\delta)))$ vanishes identically. In fact, let us suppose in the first place, that $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ is an even characteristic. Then, if $\bar{\theta}[g;h]((\bar{u}_\sigma(\delta))) \neq 0$, we deduce from (27″)

$$\exp[h_1\pi i] + \exp[-h_1\pi i] = 0;$$

hence $h_1 = \frac{1}{2}$. It follows that $[g_1, \cdots, g_p; h_1, \cdots, h_p] = [\frac{1}{2}, g_2, \cdots, g_p; \frac{1}{2}, h_2, \cdots, h_p]$ is an odd characteristic. We exclude naturally this case, because then $\theta[g;h]((0))$ vanishes identically, as function of the branch points. Likewise if $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ is an odd characteristic, we have

$$\exp[h_1\pi i] - \exp[-h_1\pi i] = 0;$$

hence $h_1 = 0$, and $[g_1, \cdots, g_p; h_1, \cdots, h_p] = [\frac{1}{2}, g_2, \cdots, g_p; 0, h_2, \cdots, h_p]$ is again an odd characteristic.

In conclusion, we have the following result: *The function* $\theta[g;h]((0))$ *possesses a zero of order* g_1^2 *at* $k = k_1$, *if* $g_1 \leqq \frac{1}{2}$, *and of order* $(1-g_1)^2$, *if* $g_1 \geqq \frac{1}{2}$. *The only exception corresponds to the case in which* $g_1 = 0$ *or* $\frac{1}{2}$ *and* $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ *is a characteristic* $[\epsilon_2]$ *such that* $\bar{\theta}[\epsilon]_2((m\bar{u}_\sigma(\delta)))$ *vanishes identically. Here* $m = 1$, *if* $g_1 = \frac{1}{2}$, *and* $m = 0$, *if* $g_1 = 0$.

We examine now this exceptional case.

6. *Examination of the exceptional case.* The values of the integrals $\bar{u}_\sigma$ $(\sigma = 2, 3, \cdots, p)$ at the branch point k_i $(i = 2, 3, \cdots, 2p+1)$ of the curve $\bar{C}$ constitute a system $\{\bar{a}_i\}$ of half periods. It is easily seen that the characteristic $(\bar{a}_i)$ is obtained from the characteristic (a_i) of the table (11) (§ 3) suppressing the two numbers of the first column. Every characteristic $[g_2, \cdots, g_p; h_2, \cdots, h_p] = [\epsilon]_2$ is of the form

$$[\bar{\kappa} + \sum^{p-1-s} \bar{a}_i],$$

where $[\bar{\kappa}] = [1^{p-1}; p-1, p-2, \cdots, 1]$, s is one of the members $0, 1, \cdots, p-1$, and $\sum^{p-1-s} \bar{a}_i$ indicates the sum of $p-1-s$ conveniently chosen characteristics $(\bar{a}_i)$.

We recall the following well-known theorems, with reference to the curve C of genus p: *

I. Indicating by $\sum^{p-s} a_i$ the sum of any $p-s$ characteristics of the table (11), the function $\theta[\kappa + \sum^{p-s} a_i]((mu_\sigma(\eta)))$ vanishes identically, if m is one of the integers $0, 1, \cdots, s-1$. However $\theta[\kappa + \sum^{p-s} a_i]((su_\sigma(\eta)))$ does not vanish identically.

* See (Kr.), p. 459.

II. Denoting by $[(s-1)/2]$ the integral part of $(s-1)/2$, the function $\theta[\kappa+\sum^{p-s} a_i]((v))$ and all its partial derivatives of order 1, 2, $\cdots$, $[(s-1)/2]$, but not all of its partial derivatives of order $[(s+1)/2]$, vanish for the zero values of the arguments $v_1, v_2, \cdots, v_p$.

Let us consider now the function $\theta[g;h]((0))$, in which $g_1 = 0$ or $\frac{1}{2}$, h_1 is arbitrary, and $[g_2, \cdots, g_p; h_2, \cdots, h_p]$ is a characteristic $[\epsilon]_2$:

$$[g_2, \cdots, g_p; h_2, \cdots, h_p] = [\epsilon]_2 = [\bar{\kappa} + \sum^{p-1-s} \bar{a}_i].$$

Looking at (24) and (25) and applying the theorem I to the curve $\bar{C}$ of genus $p-1$, we find:

1) If $g_1 = 0$, then

$$\lim_{k \to k_1} A_{m_1} = 0 \text{ for } m_1 = 0, \pm 1, \cdots, \pm [(s-1)/2],$$

and $$\lim_{k \to k_1} A_{[(s+1)/2]} \neq 0.$$

2) If $g_1 = \frac{1}{2}$, then

$$\lim_{k \to k_1} A_{m_1} = 0 \text{ for } m_1 = 0, \pm 1, \cdots, \pm [(s-2)/2], -[s/2],$$

and $$\lim_{k \to k_1} A_{[s/2]} \neq 0.$$

The characteristic $[\bar{\kappa} + \sum^{p-1-s} \bar{a}_i]$ coincides evidently with the last $p-1$ columns of the corresponding characteristic $[\kappa + \sum^{p-1-s} a_i]$. On the other hand, the addition of the characteristic (a), or of (a_1), or of their sum does not alter the $p-1$ last columns. We obtain thus the 4 characteristics

$$[\lambda, \epsilon_1, \cdots, \epsilon_p; \mu, \epsilon_2', \cdots, \epsilon_p']_2 \ (\lambda, \mu = 0, 1),$$

and, it is seen at once, that

$$[\kappa + a + \sum^{p-1-s} a_i], \quad [\kappa + a_1 + \sum^{p-1-s} a_i]$$

coincide, to within order, with the characteristics

$$[0, \epsilon_1, \cdots, \epsilon_p; 0, \epsilon_2', \cdots, \epsilon_p']_2, \quad [0, \epsilon_2, \cdots, \epsilon_p; 1, \epsilon_2', \cdots, \epsilon_p']_2,$$

while

$$[\kappa + \sum^{p-1-s} a_i], \quad [\kappa + a + a_1 + \sum^{p-1-s} a_i]$$

coincide, to within order, with

$$[1, \epsilon_2, \cdots, \epsilon_p; 0, \epsilon_2', \cdots, \epsilon_p']_2, [1, \epsilon_2, \cdots, \epsilon_p; 1, \epsilon_2', \cdots, \epsilon_p']_2.$$

We consider separately the two cases: 1) $g_1 = 0$, 2) $g_1 = \frac{1}{2}$.

1) Let $g_1 = 0$. We have

$$\theta[g;h]((0)) = \sum_{-\infty}^{+\infty} \exp\,[2m_1h_1\pi i + a_{11}m_1^2]\cdot A_{m_1}. \tag{28}$$

We observe that A_{m_1} does not depend upon g_1, h_1. Applying the theorem II to the functions

$$\theta[\kappa + a + \sum^{p-1-s} a_i]((v)), \qquad \theta[\kappa + a_1 + \sum^{p-1-s} a_i]((v))$$

and to their derivatives taken with respect to v_1, we find

$$\left.\begin{array}{l} \sum_{-\infty}^{+\infty} m_1{}^r \exp\,[a_{11}m_1^2]\cdot A_{m_1} = 0; \\ \sum_{-\infty}^{+\infty} m_1{}^r \exp\,[m_1\pi i + a_{11}m_1^2]\cdot A_{m_1} = 0; \end{array}\right\} \quad (r = 0, 1, \cdots, [(s-1)/2]. \tag{29}$$

We put for brevity $[(s-1)/2] = q$. The $2q+2$ equations (29) can be solved with respect

$$A_0,\ \exp\,[a_{11}]\cdot A_{\pm 1}, \ldots,\ \exp\,[a_{11}q_1^2]\cdot A_{\pm q},\ \exp\,[a_{11}(q+1)^2]\cdot A_{-(q+1)},$$

the corresponding determinant being different from zero. Then (28) becomes

$$\theta[g;h]((0)) = c_1\ \exp\,[a_{11}(q+1)^2]\cdot A_{(q+1)} + \exp\,[a_{11}(q+2)^2]\cdot\{c_2 A_{(q+2)} + c_{-2}A_{-(q+2)}\} + \cdots.$$

We saw above that $\lim_{k\to k_1} A_{(q+1)} \neq 0$. Hence, provided the coefficient c_1 is not 0, we have

$$\theta[g;h]((0)) \sim (k-k_1)^{(q+1)^2}. \tag{30}$$

To calculate c_1 in the simplest way, we consider the function $(e^z - e^{-z})^{q+1}$. Let

$$(e^z - e^{-z})^{q+1} = \sum_{-(q+1)}^{+(q+1)} \alpha_m e^{mz}. \tag{31}$$

Observing that $(e^z - e^{-z})^{q+1}$ and all its derivatives of order $1, 2, \cdots, q$ vanish for $z = 0$, or $z = \pi i$, we obtain the following equations:

$$\left.\begin{array}{l} \sum_{-(q+1)}^{+(q+1)} m^r\alpha_m = 0; \\ \sum_{-(q+1)}^{+(q+1)} m^r\cdot e^{m\pi i}\alpha_m = 0; \end{array}\right\} \quad (r = 0, 1, \cdots, q). \tag{32}$$

The system of equations (32) does not differ from the system (29), if in the

last one we neglect the terms of order $m_1 > q+1$, or $< -(q+1)$. Hence, comparing (31) with (28), we deduce,

$$c_1 = (e^{2h_1\pi i} - e^{-2h_1\pi i})^{q+1};$$

whence $c_1 = 0$, since evidently we must suppose $h_1 \neq 0, \frac{1}{2}$. We may write the result as follows:

$$(33)\quad \lim_{k\to k_1} \theta[g;h]((0)) \exp[-a_{11}(q+1)^2] = c_1\bar{\theta}[\epsilon]_2(((2q+2)\bar{u}_\sigma(\delta))).$$

2) Let $g_1 = \frac{1}{2}$. We have now

$$(34)\quad \theta[g;h]((0)) = \sum_{-\infty}^{+\infty m_1} \exp[(2m_1+1)h_1\pi i + a_{11}(m_1+\tfrac{1}{2})^2]\cdot A_{m_1}.$$

We apply the theorem II to the functions

$$\theta[\kappa + \sum^{p-1-s} a_i]((v)), \quad \theta[\kappa + a + a_1 + \sum^{p-1-s} a_i]((v)),$$

of which the first vanishes for $v=0$ together with its partial derivatives of order 1, 2, $\cdots$, $[s/2]$, and the second vanishes for $v=0$ together with its partial derivatives of order 1, 2, $\cdots$, $[(s-2)/2]$. It is easily seen that if s is odd, $[\kappa + \sum^{p-1-s} a_i]$ is the characteristic

$$[1, \epsilon_1, \cdots, \epsilon_p; 0, \epsilon_2', \cdots, \epsilon_p']_2,$$

and if s is even, it is the characteristic $[1, \epsilon_2, \cdots, \epsilon_p; 1, \epsilon_2', \cdots, \epsilon_p']_2$. Considering only the derivatives of both functions with respect to v_1, and putting, for brevity, $q = [(s-2)/2]$, we have in the first place the relations

$$(35)\quad \left.\begin{aligned} &\sum_{-\infty}^{+\infty m_1} (2m_1+1)^r \exp[(m_1+\tfrac{1}{2})\pi i + a_{11}(m_1+\tfrac{1}{2})^2]\cdot A_{m_1} = 0;\\ &\sum_{-\infty}^{+\infty m_1} (2m_1+1)^r \exp[a_{11}(m_1+\tfrac{1}{2})^2]\cdot A_{m_1} = 0; \end{aligned}\right\}$$

$$(r = 0, 1, \cdots, q).$$

One more equation arises from the vanishing of

$$\frac{\partial^{q+1}}{\partial v_1^{q+1}} \theta[\kappa + \sum^{p-1-s} a_i]((v))$$

for $v=0$. Hence, if s is odd, we will have

$$(35')\quad \sum_{-\infty}^{+\infty m_1} (2m_1+1)^{q+1} \exp[a_{11}(m_1+\tfrac{1}{2})^2]\cdot A_{m_1} = 0,$$

and if s is even, we will have

$$(35'')\quad \sum_{-\infty}^{+\infty m_1} (2m_1+1)^{q+1} \exp[(m_1+\tfrac{1}{2})\pi i + a_{11}(m_1+\tfrac{1}{2})^2]\cdot A_{m_1} = 0.$$

In both cases we have a system of $2q+3$ equations, which we can solve with respect to the $2q+3$ functions $\exp[a_{11}(m_1+\frac{1}{2})^2]\cdot A_{m_1}$, $[m_1=0, \pm 1, \cdots, \pm q, -(q+1), -(q+2)]$, and which permit of expressing $\theta[g;h]((0))$ in (34) by means of the remaining functions. We have then

$$\theta[g;h]((0)) = c_1 \exp[a_{11}(q+\tfrac{3}{2})^2]\cdot A_{q+1} + \cdots,$$

where the missing terms contain powers of $e^{a_{11}}$ higher than $(q+\frac{3}{2})^2$. We saw above that $\lim_{k\to k_1} A_{q+1} \neq 0$. Hence

$$\theta[g;h]((0)) \sim (k-k_1)^{(q+3/2)^2}, \tag{30'}$$

provided that the coefficient c_1 is $\neq 0$. To calculate c_1 in a simple way, we consider, if, for instance, s is odd, the function

$$F(z) = (e^z - e^{-z})^{q+2}(e^z + e^{-z})^{q+1}.$$

We may write

$$F(z) = \sum_{-(q+2)}^{(q+1)}{}_m \alpha_m e^{(2m+1)z}. \tag{36}$$

We express the fact that $F(z)$ and all its derivatives of order $1, 2, \cdots, q$ vanish for $z=0$ or $\pi i/2$, and that moreover the derivative of order $q+1$ vanishes for $z=0$. We have thus the following systems of $2q+3$ equations:

$$\sum_{-(q+2)}^{q+1}{}_m (2m+1)^r \alpha_m = 0 \qquad (r=0, 1, \cdots, q+1)$$

$$\sum_{-(q+2)}^{q+1}{}_m (2m+1)^r \exp[(m+\tfrac{1}{2})\pi i]\cdot \alpha_m = 0 \qquad (r=0, 1, \cdots, q).$$

Comparing this system with the system of equations (35), (35′) in which we may neglect all terms of order $m_1 > (q+1)$ or $< -(q+2)$, since we need only the coefficient of $\exp[a_{11}(q+\frac{3}{2})^2]A_{q+1}$, we notice that these two systems of linear equations possess the same coefficients. Hence, comparing (34) and (36), we deduce

$$c_1 = (e^{h_1\pi i} - e^{-h_1\pi i})^{q+2}(e^{h\pi_1 i} + e^{-h_1\pi i})^{q+1}.$$

If $h_1 \neq 0, \frac{1}{2}$, then $c_1 \neq 0$.

If s is even, we consider the function

$$(e^z - e^{-z})^{q+1}(e^z + e^{-z})^{q+2},$$

and we deduce in the same way

$$c_1 = (e^{h_1\pi i} - e^{-h_1\pi i})^{q+1}(e^{h_1\pi i} + e^{-h_1\pi i})^{q+2}.$$

We may express the result as follows:

$$(33') \qquad \lim\ \theta[g;h]((0))\exp[-a_{11}(q+\tfrac{3}{2})^2] = c_1\bar{\theta}[\epsilon]_2(((2q+3)\bar{u}_\sigma(\delta_2))),$$

where

$$c_1 = (e^{h_1\pi i} + (-1)^s e^{-h_1\pi i})^{q+2}(e^{h_1\pi i} - (-1)^s e^{-h_1\pi i})^{q+1}$$

and $q = [(s-2)/2]$.

7. *The degree of the invariant I_r.* We return to the formula (17) of section 4, and put

$$(37) \qquad F_r(k, k_1, \cdots, k_{2p+1}) = D^h I_r^2,$$

where D is the discriminant of the polynomial $f(x)$ and I_r is the corresponding invariant of $f(x)$. We write I_r^2, because if $\theta[g;h]((0)) = 0$, then also $\theta[-g;-h]((0)) = 0$, and generally the vanishing of $\theta[g;h]((0))$ does not imply the vanishing of any other θ-function. From the algebraic point of view this means that if the polynomials satisfies the relation (10) or (10'), then generally the polynomials ϕ_{m-p-1} or ϕ_{m-p}, ψ_m and ψ_{p-1} are uniquely determined.*

Moreover we observe that F_r possesses a finite value for finite values of the k_i's. In fact, let us fix $k_1, \cdots, k_{2p+1}$. If k does not coincide with any k_i, then every θ-function of the product has a finite value, while the determinant $|A_{ij}|$ is $\neq 0$. Suppose now $k = k_i$; due to symmetry we may suppose $k = k_1$. Every function θ of the product possesses still a finite value. As far as the determinant $|A_{ij}|$ is concerned, its limit for $k = k_1$ can be calculated by using Thomae's formula (12), taking into consideration that $\lim_{k\to k_1} \theta((0)) = \bar{\theta}((0))$ (see § 5). One finds then

$$\lim_{k\to k_1} |A_{ij}| = 2\pi i\ |\bar{A}_{\sigma\mu}|/\{\prod_2^{2p+1}{}_i (k_1 - k_i)\}^{\frac{1}{2}},$$

where the $\bar{A}_{\sigma\mu}$'s are the analogous periods relative to the limit curve $\bar{C}$. Hence this limit is still $\neq 0$, and therefore F_r is finite for $k = k_1$. This proves the correctness of (37).

---

* Take for instance the case $p = 5$, $r = 3$, i. e. a polynomial $f(x)$ of degree 12 and of the form $f(x) = j^3 + i^3$. Suppose we have another representation of the same polynomial: $f(x) = j_1^3 + i_1^3$. Then $i_1^3 - i^3 = (j - j_1)(j + j_1)$. If we fix i_1 and i (and this furnishes 8 arbitrary constants and one constant more as factor of proportionality in $i_1^3 - i^3$) we have ∞^1 solutions for j and j_1. It follows that the polynomials $f(x)$ which admit two different representations of the form $j^3 + i^3$ depend upon 10 arbitrary constants, while the general polynomial of this form depends upon 11 constants.

By means of the results of sections 5 and 6 we are able to evaluate the exponent h. In fact $|A_{ij}|$ does not vanish for $k \to k_1$, hence F_r vanishes for $k \to k_1$ to the same order, namely $2h$, as the product of the function θ. Let h be calculated. From the formula of Thomae we deduce that $|A_{ij}|^{\frac{1}{2}}$ vanishes at $k = \infty$ to order $p/4$, because, at $k = \infty$, $\theta((0))$ is $\neq 0$, while the radical which appears in the second member becomes ∞ to order $p/4$. Since every function $\theta[\kappa + \{\epsilon\}_r]((0))$ possesses generally for $k = \infty$ a value different from 0, it follows that F_r becomes ∞ at $k = \infty$ to order $N_r' p/4$, where N_r' is the number defined in section 4. Since the discriminant D is of degree $2(2p+1)$ in k, we deduce, denoting by g_r the degree of I_r:

$$2g_r = N_r' p/4 - 2h(2p+1). \tag{38}$$

Let us calculate h and then g_r in the case of p odd and r an odd prime number. There is no difficulty in calculating h in the most general case. Since r is a prime number and $r(p-1)$ is even, we have $N_r' = N_r$ and from (13)

$$N_r' = N_r = r^{2p} - 1. \tag{39}$$

The $r^{2p} - 1$ functions θ of the product are in this case all functions

$$\theta[\kappa + \{\epsilon\}_r]((0)), \tag{40}$$

in which, however, the characteristic has to be reduced modulo $2r$.

Let us denote the reduced characteristic by

$$[\eta]_{2r} = [\eta_1, \cdots, \eta_p; \eta_1', \cdots, \eta_p']_{2r}.$$

Here every η_i is one of the numbers $(r-2s)/2r$, $(r+2s)/2r$, where $s = 0, 1, \cdots, (r-1)/2$. All functions (40) for which $\eta_1 = (r-2s)/2r$ or $= (r+2s)/2r$ possess, for given $s \neq 0$, at $k = k_1$ a zero of order $(r-2s)^2/(2r)^2$ (see (26) and (26′), section 5). The number of such functions is equal to $2r^{2p-1}$, hence their product possesses at $k = k_1$ a zero of order

$$2r^{2p-1}(r-2s)^2/(2r)^2 = r^{2p-3}(r-2s)^2/2.$$

Giving to s the values 1, 2, $\cdots$, $(r-1)/2$, we deduce that the product of all those functions (40), for which η_1 is $\neq \frac{1}{2}$, possess at $k = k_1$ a zero of order

$$(r^{2p-3}/2)\{1^2 + 3^2 + \cdots + (r-2)^2\} = r^{2p-2}(r-1)(r-2)/12. \tag{41}$$

We consider now those functions (40) in which $\eta_1 = \frac{1}{2}$. These possess at $k = k_1$ a zero of order $\frac{1}{4}$, unless the characteristic $[\eta_2, \cdots; \eta_2', \cdots]_{2r}$ is the characteristic $[\tilde{\kappa}]_2$, in which case we have special functions considered in section 6. The number of these special functions is $r-1$, and are obtained by giving to ϵ_1' the values 1, 2, $\cdots$, $r-1$. Hence the number

of the remaining non-special function is $r^{2p-1} - r$, and their product possesses at $k = k_1$ a zero of order

(42) $$(r^{2p-1} - r)/4.$$

Finally every function

$$\theta[\kappa + \{0^p; \epsilon_1', 0^{p-1}\}_r]((0))$$

possesses at $k = k_1$ a zero of order $p^2/4$, as a result of (30′), since in this case $s = p - 1$, hence $q = (p-3)/2$. The product of these $r - 1$ functions possesses then at $k = k_1$ a zero of order

(43) $$(r-1)p^2/4.$$

Adding the orders written in (41), (42), (43) we deduce that the product of all $r^{2p} - 1$ functions (40), and hence F_r, possesses at $k = k_1$ a zero of order

$$r^{2p-2}(r-1)(r-2)/12 + (r^{2p-1} - r)/4 + (r-1)p^2/4 =$$
$$= (r^2+2)r^{2p-2}/12 + (r-1)p^2/4 - r/4.$$

Hence this is the value of $2h$. Substituting this value of $2h$ and the value of N_r', given by (39), in (38) we find:

(44) $$2g_r = r^{2p-2}(pr^2 - r^2 - 4p - 2)/12$$
$$- [(r-1)p^2 - r](2p+1)/4 - p/4.$$

For $p = 5$ and $r = 3$ we find the degree of the invariant connected with the 12 tangents to a plane quartic curve. The degree of this invariant is equal

$$2 \cdot 3^2 \cdot 11 \cdot 19.$$

THE JOHNS HOPKINS UNIVERSITY.

Certain Perfect Groups Generated by Two Operators of Orders Two and Three.

By H. R. Brahana.

Introduction

We have shown recently * that there is a one-to-one correspondence between the regular maps and groups generated by two operators of which one is of order two. Certain classes of groups came immediately to our attention, the symmetric and alternating groups, the meta-cyclic groups, and the linear fractional groups, $LF(2, p)$. Naturally one seeks other groups or classes of groups satisfying the given condition. Prof. Miller has shown that there exists an infinite number of finite groups generated by S and T satisfying the relation $T^2 = S^k = (ST)^l = 1$ for every pair of numbers k and l both greater than 3 or one equal to 3 and the other greater than 5. We undertook to examine some of the groups for $k = 3$ and $l = 7$. The immediate result was the series of theorems in § 1, concerned mostly with perfect groups. For want of a better method we decided to fix the order of the commutators of the generating elements and examine the resulting groups or classes of groups. The result (§ 2) was the discovery that if the order were fixed at 4 the simple group of order 168 is determined, and if fixed at 6 the simple group of order 1092 is determined, no other number less than 7 being possible. The work required by our method of procedure is tedious and promises to become more so as the order of the commutator is increased. Rather than continue in this manner we have taken the simple group of order 504 and found that it can be generated by two operators of orders two and three with product and commutator of orders seven and nine. We ask whether these conditions are sufficient to determine the group. We are unable to answer the question definitely, but we obtain sets of relations defining abstractly the generators of this group and also of the simple group of order 660. In each case we have been obliged to insert a condition in addition to fixing the orders of the generators, their product, and their commutator, without being able to prove that it is independent.

1. *The groups generated by S and T satisfying the relations* $S^3 = T^2 =$

* "Regular Maps and their Groups," *American Journal of Mathematics*, Vol. 49 (1927), pp. 266-284.

$(ST)^n = 1$. We consider two operators S and T satisfying the relations $S^3 = T^2 = (ST)^n = 1$, where each exponent is the smallest number for which the operator to which it is attached becomes identity. The group G, determined by these operators is completely determined if n is one of the numbers 3, 4, or 5.* On the other hand, if n is greater than 5 there exists an infinite number of finite groups each generated by an S and a T satisfying the given relations.† The groups for which $n = 6$ have been described in detail.‡ It is our purpose to exhibit some of the properties of the group G for certain types of value of n greater than 6.

The commutators of T and powers of S are $\sigma_1 = TS^2TS$, $\sigma_2 = TSTS^2$ and their inverses. The commutator subgroup H of G is generated by σ_1 and σ_2.§ The following relations exist among S, T and the σ's: $T\sigma_i = \sigma_i^{-1}T$, $S\sigma_1 = \sigma_2^{-1}S$, and $S\sigma_2 = S \cdot TS\,TS^2 = STS^2T \cdot TS^2TS \cdot S = \sigma_2^{-1}\sigma_1 S$.

Any element of G is of the form $S^\alpha TS^\beta T\, S^\gamma T\, S^\delta \cdots$ where $\alpha = 0$, 1, or 2, and each of the other exponents is 1 or 2. If we rewrite the above product in the form $S^\alpha\,(TS^\beta TS^{-\beta})\, S^{\beta+\gamma}\,(TS^\delta TS^{-\delta})\, S^{\delta+\epsilon} \cdots$ and make use of the relations among S, T, and the σ's, we see that every element of G is in one of the forms

h, hS, hS^2, hT, hST, hS^2T, $hSTS$, hS^2TS, $hSTS^2$, hS^2TS^2, where h is a product of the σ's and hence an element of H. Moreover, the last four types are included among the preceding six due to the relations: $STS = STS^2T \cdot TS^2 = STS^2T \cdot TS^2TSS^2T = \sigma_2^{-1}\sigma_1 \cdot S^2T$; $S^2TS = \sigma_1^{-1} \cdot T$; $STS^2 = \sigma_2^{-1}T$; and $S^2TS^2 = S^2TST \cdot TS = S^2TST \cdot TSTS^2 \cdot ST = \sigma_1^{-1}\sigma_2 \cdot ST$. These six types are distinct provided neither of the generators S and T is in the commutator subgroup of G.

We have the following theorem:

The order of G is at most six times the order of its commutator subgroup.

The successive powers of (ST) are:

$$\begin{aligned} ST &= ST \\ (ST)^2 &= \sigma_2^{-1}\sigma_1 \cdot S^2 \\ (ST)^3 &= \sigma_2^{-1}\sigma_1 \cdot T \\ (ST)^4 &= \sigma_2^{-1}\sigma_1\sigma_2 \cdot S \end{aligned}$$

---

* Cf. Burnside, *Theory of Groups* (1897), p. 291.

† Miller, *American Journal of Mathematics*, Vol. 24 (1902), pp. 96-100.

‡ Miller, *Quarterly Journal of Mathematics*, Vol. 33 (1901), pp. 76-79.

§ Miller, "On the Commutator Groups," *Bulletin of the American Mathematical Society*, Vol. 4 (1897), pp. 135-139.

$$(ST)^5 = \sigma_2^{-1}\sigma_1\sigma_2 \cdot S^2T$$
$$(ST)^6 = \sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}$$
$$(ST)^7 = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})\,(ST)$$
$$\vdots$$
$$(ST)^{6k+l} = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1)^k \cdot (ST)^l.$$

From the above relations we see that if n, the order of (ST), is 6 the commutator sub-group of G is Abelian.* Moreover, we may state the following converse theorem:

If two non-commutative operators of orders two and three generate a group whose commutator sub-group is Abelian their product is of order six.

If (ST) is of order 7, then $(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})\,(ST) = 1$ and therefore (ST) is in H. If (ST) is in H we see from the values of $(ST)^3$ and $(ST)^4$ that both S and T are in H. Likewise S and T are in H if the order of (ST) is any number of the form $6k+1$. Moreover, if the order of (ST) is of the form $6k-1$ S^2T is in H, and since $S^2T \cdot ST = \sigma_1^{-1}$, (ST) and, consequently, S and T are in H. Hence, G and H coincide. Such a group is called perfect.† We have the following theorem:

Any group generated by two operators of orders two and three respectively the order of whose product is of the form $6k \pm 1$ is perfect.

We have, *a fortiori,* the theorem:

Two operators of orders two and three whose product is of prime order greater than three generate a perfect group.

Every perfect group is simple or else it is isomorphic $(\alpha - 1)$ to some simple group of composite order.‡ Hence,

No group generated by two operators of orders two and three the order of whose product is a prime $6k \pm 1$ can have an order less than that of the smallest simple group whose order contains the factors 2, 3, and $6k \pm 1$.

For, the simple group to which G is isomorphic will be generated by two operators which correspond in the isomorphism to the generators of G and therefore must be or orders two and three with product of order $6k \pm 1$.

---

* This fact was used by Prof. Miller in studying the groups in question. Cf. the reference above, σ_1 and σ_2 are the same as his s_3 and s_4.

† Miller, *American Journal of Mathematics,* Vol. 20 (1898), pp. 277. See footnote.

‡ Miller: Cf. the last reference.

We may also state the following theorem:

In a group whose order is not a multiple of the order of some simple group it is impossible to select an operator of order two and an operator of order three such that their product is of order $6k \pm 1$.

If the order of (ST) is of the form $6k \pm 2$, S is in H; and if of the form $6k + 3$, T is in H. Hence,

If a group generated by two operators of orders two and three whose product is of order $6k \pm 2$ *(or* $6k + 3$*) its order is at most* 2 *(or* 3*) times the order of its commutator subgroup.*

The transform of σ_2 by (ST) is $TS^2 \cdot TSTS^2 \cdot ST = TS^2TS = \sigma_1$ and therefore σ_1 and σ_2 are of the same order. Hence,

Any group generated by two operators of orders two and three whose product is of order $6k \pm 1$ *is also generated by two operators of equal orders.*

If the order of ST is of the form $6k$ the subgroup H may or may not contain either or both of the operators S and T. If it contains neither so that the sets H, HS, HS^2, HT, HST, and HS^2T are all distinct, it follows that the quotient group G/H is cyclic. As a result of this and the foregoing theorems we have the theorem:

The largest Abelian group to which a group generated by two operators of orders two and three is isomorphic is a cyclic group of order 1, 2, 3, *or* 6, *according as the order of the product of the generators is equivalent to* 1, 2, 3, *or* 0, *mod* 6.

From the value of $(ST)^7$ above we see that if $(ST)^7 = 1$ (ST) is the commutator of σ_1^{-1} and σ_2. Hence,

In any group generated by T *and* S *of orders two and three such that* ST *is of order* 7, ST *is the commutator of the commutators of* T *and* S *and of* S^2 *and* T.

2. *Relations defining generators of certain simple groups.* Any group G generated by two operators S and T of orders 3 and 2 whose product is of order 7 contains two operators σ_1 and σ_2 which also serve to generate the group. The elements σ_1 and σ_2 satisfy the following conditions:

(A) $(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^7 = 1$;

(B) $(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^3\sigma_2^{-1}\sigma_1 = \sigma_2\sigma_1^{-1}(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^3$;

(C) $(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^2\sigma_2^{-1}\sigma_1 \cdot (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^2 = \sigma_1^{-1}\sigma_2(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^5\sigma_1^{-1}\sigma_2$;

(D) $\sigma_1 = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^2\sigma_1^{-1}\sigma_2(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^2$;

Conversely, *any group G generated by two operators σ_1 and σ_2 satisfying (A), (B), (C), and (D) is also generated by two operators of orders two and three whose product is of order 7.*

For, the element $T' = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^3\sigma_2^{-1}\sigma_1$, is of order two as a result of (B); the element $S' = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^2\sigma_2^{-1}\sigma_1$ is of order three as a result of (C); their product, $S'T' = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^6$, is of order 7 by (A); the group $[S', T']$ contains σ_1 as a result of (D); and $[S', T']$ contains σ_2 since it contains $T', S'T'$, and σ_1.

The above theorem states the sufficiency of the given conditions on σ_1 and σ_2 in order that the group be of the type in question. The conditions are of course necessary as may be seen by consideration of the group $[S, T]$ in the manner of § 1, but there arises the question of their independence, whether or not one or more of them may not be omitted. In attempting to answer the question two methods of procedure are possible. First, the independence would be shown by giving examples satisfying all but one of the conditions and contradicting the one; or, on the other hand, the redundancy of the conditions might be shown by deriving a certain number of them on the basis of the others. Both attempts have been made with no result other than a slight reflection on the ingenuity of the author. However, we may arrive at a plausible answer in the following manner: Conditions (B) and (C) insure the presence in $[\sigma_1, \sigma_2]$ of operators of orders two and three, condition (A) requires one pair of such operators to have a product of order 7, and condition (D) requires the group generated by the two operators just described to be the whole group.

In investigating the groups under consideration we make use of the co-sets with respect to a given subgroup, which should be as large as can be conveniently arranged so that the number of co-sets will be small, and which should be designated by simple expressions in the generators. For this reason we determine another set of generators of G.

Let $(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})$ be denoted by R and let $\sigma_2^{-1}\sigma_1$ be denoted by Q. Then $S' = R^2Q$ and $T' = Q^{-1}R^4$. Relations (A), (B), (C), and (D) become

$$(A') \quad R^7 = 1;$$

$$(B') \quad Q^{-1}R^4 = R^3Q;$$

$$(C') \quad R^2QR^{-1} = Q^{-1}RQ.$$

Condition (D) expressed in terms of Q and R is a consequence of (A'),

(B'), (C'), for if $S' = R^2Q$ and $T' = R^3Q$, $[S', T']$ contains both Q and R. This does not say, however, that condition (D) is a consequence of the three conditions which precede it, for presumably σ_1 and σ_2 might generate a group of which the group generated by Q and R or by S' and T' would be a subgroup.

Any group generated by two operators Q and R satisfying relations (A'), (B'), and (C'), is also generated by two operators of orders two and three, whose product is of order seven.

We shall now fix the order of Q and examine the resulting groups. First we observe that Q cannot be of order 2 for then (B′) would become $QR^4 = R^3Q$, whence $QR^4Q = R^3$ which requires the group to be the dihedral group of order 14. The dihedral group of order 14 contains no operator of order 3 and therefore condition (C′) cannot be satisfied. Hence,

There is no group generated by two operators of orders two and three whose product is of order 7 and whose commutator is of order 2.

Let us suppose that Q is of order 3. Then by (C) we have $QR^2 = RQRQ^2$. Applying this relation to $QRQR^2$ we get

$$\begin{aligned} QRQR^2 &= QR^2QRQ^2 \\ &= R(QR^2Q)Q \\ &= R(RQR)Q \\ &= R^2QRQ. \end{aligned}$$

This states that R^2 is permutable with QRQ as is also $(R^2)^4 = R$. As a result of this fact $QR^2 = RQRQ^2$ gives

$$QR^2 = R(QRQ)Q = QRQRQ$$

whence $R = QRQ$. This result applied to (B′) gives

$$\begin{aligned} R^4 &= QR^3Q \\ &= QRQ \cdot Q^2RQ^2 \cdot QRQ \\ &= R(QRQ)R = R. \end{aligned}$$

This contradicts (A'); therefore,

There is no group generated by two operators of orders two and three whose product is of order 7 and whose commutator is of order 3.

If Q is of order 4 the group is the simple group of order 168. To show this we shall show that the operators S and T satisfy the relations given by Burnside * to determine the simple group of order 168. We proceed to show that the condition $Q^4 = 1$ is equivalent under conditions (A′), (B′), and

* *Theory of Groups*, p. 422. (Second edition).

(C′) above to the condition $[(ST)^4T]^4 = 1$. The group $[Q, R]$ is generated by S and T whose commutator is of order 4. The commutator of T and S is TS^2TS, putting the fourth power of it equal to zero, we have $[TS^2TS]^4 = [TST \cdot TSTS]^4 = 1$, or

$TST \cdot TSTSTST \cdot TSTSTST \cdot TSTSTST \cdot TSTS = 1$. This implies that $(TSTSTST)^4 = 1$. Since $(ST)^7 = 1$, the last statement implies that $(STSTSTS)^4 = 1$, which is $[(ST)^4T]^4 = 1$, the condition given by Burnside. Therefore,

The group generated by two operators of orders two and three whose product is of order 7 and whose commutator is of order 4 is the simple group of order 168.

We shall proceed to show that $Q^5 = 1$ is impossible by showing that such an assumption leads, in view of relations (A′), (B′), and (C′), to the relation $QRQ = Q$.

$$QRQ = QR^4 \cdot R^4Q = R^3 \cdot R^4QR^4 \cdot R^4QR^4 \cdot R^3. \tag{1}$$

The product $R^4QR^4 = R^3 \cdot RQR \cdot R^3$ by (C′) becomes $R^3 \cdot QR^2Q \cdot R^3$. This becomes by (B′) $Q^4R^4 \cdot R^2 \cdot R^4Q^4 = Q^4R^3Q^4$, which reduces the rightmost member of (1) to

$$R^3 \cdot Q^4R^3Q^4 \cdot Q^4R^3Q^4 \cdot R^3 = R^3Q^4 \cdot R^3Q^3R^3 \cdot Q^4R^3. \tag{2}$$

The product $R^3Q^3R^3$ may be written as $R^3Q \cdot Q \cdot QR^3$ which by (B′) is $Q^4 \cdot R^4QR^4 \cdot Q^4$, which is $Q^3R^3Q^3$. Hence, the right member of (2) is

$$R^3Q^4 \cdot Q^3R^3Q^3 \cdot Q^4R^3 = R^3 \cdot Q^2R^3Q^2 \cdot R^3. \tag{3}$$

The right member of (3) may be written as

$$R^5 \cdot R^5Q^2R^5 \cdot R^5Q^2R^5 \cdot R^5. \tag{4}$$

The element $R^5Q^2R^5$ reduces to Q in the following manner.

(5) $R^5Q^2R^5 = R^3 \cdot R^2Q^2R^2 \cdot R^3.$

$$\begin{aligned}
R^2Q^2R^2 &= R^2Q \cdot QR^2Q \cdot Q^4 \\
&= R^2Q \cdot RQR \cdot Q^4 = R \cdot RQR \cdot QRQ^4 \\
&= R \cdot QR^2Q \cdot QRQ^4 = RQR \cdot RQ^2RQ^4 \\
&= QR^2Q \cdot RQ^2RQ^4 = QR \cdot RQR \cdot Q^2RQ^4 \\
&= QR \cdot QR^2Q \cdot Q^2RQ^4 = Q \cdot RQR \cdot RQ^3RQ^4 \\
&= Q^2R^2Q \cdot RQ^3R \cdot Q^4 = Q^2R \cdot RQR \cdot Q^3RQ^4 \\
&= Q^2R \cdot QR^2Q \cdot Q^3RQ^4 = Q^2RQR^2 \cdot Q^4RQ \cdot Q^3.
\end{aligned}$$

The element Q^4RQ is by (C′) R^2QR^6 and hence $R^2Q^2R^2 = Q^2RQR^2 \cdot R^2QR^6 \cdot Q = Q^2RQR^4QR^6 \cdot Q^3 = Q^2RQR^3 \cdot RQR \cdot R^5Q^3$ which by (C′) again becomes $Q^2R \cdot QR^3Q \cdot R^2QR^5Q^3$. If in this we use (B′) to replace QR^3Q by R^4 we have

$$R^2Q^2R^2 = Q^2R \cdot R^4 \cdot R^2QR^5Q^3 = Q^3R^5Q^3.$$

Using this in (5) we get

(6) $R^5Q^2R^5 = R^3 \cdot Q^3R^5Q^3 \cdot R^3$ which may be written $R^3Q^3R^3 \cdot R^6 \cdot R^3Q^3R^3$. Replacing $R^3Q^3R^3$ by $Q^3R^3Q^3$ we get

(7) $R^5Q^2R^5 = Q^3R^3Q^3 \cdot R^6 \cdot Q^3R^3Q^3 = Q^3 \cdot R^3Q^3R^3 \cdot R^3Q^3R^3 \cdot Q^3$.

By means of the same substitution we get

(8) $R^5Q^2R^5 = Q^3 \cdot Q^3R^3Q^3 \cdot Q^3R^3Q^3 \cdot Q^3 = QR^3QR^3Q$, which by (B′) is Q.

Using this in (4) we get $QRQ = R^5 \cdot Q \cdot Q \cdot R^5 = Q$. Hence, $QR = 1$ which is impossible. Therefore,

There is no group generated by two operators of orders two and three whose product is of order 7 and whose commutator is of order 5.

If Q is of order 6 it turns out that the group is of order 1092. Our method was to determine the number of right co-sets with respect to the cyclic group generated by R. There are 156 such sets counting the group generated by R. Each set may be designated by a single product of Q's and R's where the first element in the product is a power of Q; for example, QR may be allowed to stand for the set $QR, RQR, \cdots, R^6QR$. To write the resulting 156 operators would require about two printed pages, so we shall adopt a notation to allow us to cut down the space required. We shall denote each co-set, whose symbol ends in R, by a letter, for example b is the co-set QR. For every set whose symbol ends in R there are five more sets obtained from it by multiplication on the right by powers of Q. For each such set we shall use the same letter as for the first one except that we shall affect it with a subscript equal to the index of the power of Q on the right. Thus, b_1 stands for QRQ, b_2 for QRQ^2. We shall not stop to prove that b_i and b_j are distinct, a fact which will be evident when we obtain the final result. Thus we shall be able to indicate the 156 co-sets by giving 26 of them.

The 156 co-sets are $a, a_1, a_2, a_3, a_4, a_5, b, b_1, b_2, b_3, b_4, b_5, \cdots, z, z_1, z_2, z_3, z_4, z_5$, where

$a = 1$ $\quad j = Q^3R$ $\quad s = QR^4Q^3R^6$
$b = QR$ $\quad k = Q^3R^2$ $\quad t = QR^4Q^4R$
$c = QR^4$ $\quad l = Q^3R^5$ $\quad u = QR^4Q^4R^2$
$d = QR^5$ $\quad m = Q^3R^6$ $\quad v = QR^5Q^2R^5$
$e = Q^2R$ $\quad n = Q^4R$ $\quad w = QR^5Q^3R$
$f = Q^2R^2$ $\quad o = Q^4R^2$ $\quad x = QR^5Q^3R^2$
$g = Q^2R^4$ $\quad p = QRQ^2R^4$ $\quad y = Q^2RQ^2R^4$
$h = Q^2R^5$ $\quad q = QR^4Q^3R^2$ $\quad z = Q^2R^5Q^3R^6$
$i = Q^2R^6$ $\quad r = QR^4Q^3R^5$

Multiplication on the right permutes the co-sets in the following manner:

$$Q \equiv (aa_1a_2a_3a_4a_5)(bb_1b_2b_3b_4b_5)\cdots(zz_1z_2z_3z_4z_5);$$

$$R \equiv (a_1bb_5a_5cdc_1)(a_2efc_5ghi)(a_3jkg_5g_2lm)$$
$$(a_4nog_1c_2f_2e_5)(b_1f_1b_4d_5i_5n_1d_1)(b_2l_5e_4i_4pj_1o_1)(b_3h_5m_5p_5n_2e_1k_1)$$
$$(c_3m_1qn_5g_4rs)(c_4tug_3i_1q_2j_5)(d_2k_2l_4d_4p_4vp_1)(d_3wxp_3p_2x_4w_5)$$
$$(e_2x_3l_3m_4yw_1v_1)(e_3v_5w_4y_5j_2k_3x_1)(f_3q_1f_5o_5s_5t_1h_1)(f_4r_5j_4s_4t_2m_2u_1)$$
$$(h_2u_2r_4o_4s_3v_3t_3)(h_3w_3r_2s_2y_2k_4z)(h_4i_3y_5y_1n_3o_2x_2)(i_2s_1q_4t_5n_4z_1z_3)$$
$$(j_3v_2z_5v_4m_3u_3r_3)(k_5u_5z_2r_1l_1q_3q_5)(l_2y_3t_4u_4w_2o_3z_4)(a)(x_5).$$

The work required to show that the sets are permuted by R in the manner described above had to be done, but it is obviously too long to be given here. Each step was similar to, though fortunately not so long and involved as, the work done to show that Q could not be of order 5.

Assuming the correctness of the expression for R, all that is required to prove that the group $[Q, R]$ is of order 1092 is to make certain that the substitution group $[Q, R]$ is transitive. The transitivity follows from the facts that Q permutes the different subscripts on each letter cyclically and that in the first four cycles of R we have a with every letter up to o while every cycle of R begins with a letter which precedes o in the alphabet.

The group $[Q, R]$ is perfect and is of order 1092. If a perfect group contains an invariant subgroup the corresponding quotient group is simple and of composite order. Since there exists no simple group whose order is less than 1092 and is contained in 1092 it follows that the group $[Q, R]$ is simple. There being but one simple group of this order it is thereby completely identified. This group is described by Burnside.* We may describe the group as follows:

---

* *Proceedings of the London Mathematical Society*, Vol. 26 (1895), p. 338.

(1) *The group generated by two operators of orders two and three whose product is of order 7 and whose commutator is of order 6 is the simple group of order* 1092.

(2) *The group generated by two operators of order* 6 *satisfying conditions* (A), (B), (C), *and* (D) *is the simple group of order* 1092.

(3) *The group generated by two operators,* Q *of order* 6 *and* R, *satisfying conditions* (A'), (B'), *and* (C') *is the simple group of order* 1092.

Instead of continuing in this manner, assigning values to the order of Q and examining the resulting groups, we note that 504 contains the factors 2, 3, and 7 and inquire whether the simple group of that order may not be generated by two operators. The three operators $s = (1254673)$, $t = (12)(34)(56)(78)$, and $A = (23)(46)(57)(89)$ generate G_{504}.* We note that $s^3t = (1362587)$ and that $s^3tA = (127)(346)(598)$. If we change to our own notation by letting $s^3tA = S$ and $A = T$, we have S and T satisfying the relations $S^3 = T^2 = (ST)^7 = 1$. The commutator of S and T is $\sigma_1 = (187623459)$. The group generated by S and T is perfect, its order contains the factors 2, 7, and 9, as well as the order of some simple group, and its order is a factor of 504. The only possibility is that S and T generate the simple group of order 504. Therefore,

The simple group of order 504 *is generated by two operators of orders two and three whose product is of order 7 and whose commutator is of order* 9.

We inquire whether the converse of the above theorem is true, whether two abstract operators satisfying the above conditions necessarily generate G_{504}. We change to the generators Q and R and require Q to be of order 9.

Our answer to the question is unsatisfactory for the reason that we have found it necessary to impose a further condition on Q and R without being able to prove that the added condition is independent of the four given above. This condition is

$$(D') \qquad R^2Q^6 = Q^3R^5.$$

This condition is satisfied by the generators of the substitution group given above, but seems not to be a consequence of relations (A′), (B′), and (C′). An attempt was made to establish its independence by replacing it by the condition that the order of Q^3R^5 be four instead of two; the attempt was abandoned because it seemed to be leading to a group whose order was almost unmanageably large, at least 15,000.

---

* Burnside, *Theory of Groups* (1897), p. 375.

As before, we determine the number of right co-sets with respect to the subgroup generated by R. We find that there are 72 sets which, by means of the same notation as was used before, may be designated by a, a_1, a_2, $\cdots$, a_8, b, $\cdots$, h_8, where

$$\begin{aligned} a &= 1 & e &= Q^7R \\ b &= Q^8R & f &= Q^7R^3 \\ c &= Q^8R^2 & g &= Q^7R^5 \\ d &= Q^8R^5 & h &= Q^7R^6. \end{aligned}$$

The sets are permuted by multiplication on the right according to the substitutions

$$Q^8 \equiv (aa_1a_2a_3a_4a_5a_6a_7a_8) \cdots (hh_1h_2h_3h_4h_5h_6h_7h_8);$$

$$\begin{aligned} R \equiv\ & (a_1bcb_8a_8dd_1)(a_2ec_6fb_6gh)(a_3b_4a_6h_5f_8f_2h_3)(a_4a_5d_6g_6h_4g_5d_4) \\ & (a_7h_8g_2b_1f_1c_3e_4)(b_2f_5b_6h_2e_3e_1h_6)(b_3c_5d_7f_3h_1e_6g_4)(b_5g_7e_7h_7f_7d_3c_4) \\ & (c_1e_5e_8c_8d_8g_1d_2)(c_2g_8e_2g_3c_7f_6f_4)(a)(d_5). \end{aligned}$$

If we express (D′) in terms of S and T we may state the following theorem:

The group generated by two operators of orders two and three, whose product is of order 7 and whose commutator is of order 9 and which satisfy the further condition

$$(STSTS^2TS^2TSTS^2TS^2T)^2 = 1$$

generate the simple group of order 504.

We consider next the simple group of order 660. According to Cole * the substitution group on 11 letters is generated by $(abiegdhfcjk)$, $(aegci)(bfjhd)$, and $(ac)(ba)(eg)(fh)$. The product of the first by the last is $S = (adf)(big)(cjk)$. If we denote $(ac)(bd)(eg)(fh)$ by T, we have S and T of orders three and two with product of order 11. S and T must generate the whole group since there is no simple group of lower order whose order contains the factors 2, 3, and 11.

The commutator σ_1 is $(achif)(bjdeg)$. The simple group of order 60, the icosahedral group, is a subgroup of G_{660} and we note that $S' = (ST)^{-2} \cdot S \cdot (ST)^2 = (abk)(egh)(fji)$ and T have the product $(adbkc)(fjihg)$ and therefore must generate such a subgroup.

We inquire whether two abstract operators S and T subject to the con-

* *Quarterly Journal of Mathematics*, Vol. 27 (1894), p. 49.

ditions that they be of orders 3 and 2 with product of order 11 and commutator of order 5 necessarily generate the simple group of order 660. Again we change to operators Q and R, where R is defined as above but Q is $\sigma_1^{-1}\sigma_2$. Q and R satisfy the relations:

$$(A'') \qquad R^{11} = 1;$$
$$(B'') \qquad QR^6 = R^5Q^4;$$
$$(C'') \qquad R^7Q^4R^7 = QR^4Q;$$
$$(D'') \qquad Q^5 = 1.$$

Here again the conditions seem not to be sufficient and we have found it necessary to add the condition

$$(E'') \qquad RQ^2 = Q^3R^{10}.$$

We use the same notation as before and denote the right co-sets with respect to the group generated by R as $a, a_1, a_2, a_3, a_4, \cdots, l, l_1, l_2, l_3, l_4$, where

$$a = 1 \qquad e = QR^7 \qquad i = Q^2R^4$$
$$b = QR \qquad f = QR^8 \qquad j = Q^2R^5$$
$$c = QR^2 \qquad g = QR^{10} \qquad k = QRQ^3R^2$$
$$d = QR^3 \qquad h = Q^2R^3 \qquad l = QRQ^3R^4.$$

The corresponding substitutions on the 60 letters are

$$Q = (aa_1a_2a_3a_4) \cdots (ll_1l_2l_3l_4);$$
$$R = (a_1bcdc_4b_2a_4eff_1g)(a_2a_3g_3hijb_1j_5i_2h_3e_2)$$
$$(b_3j_3ke_1ll_1g_4k_2j_1b_4c_2)(c_1d_3i_1d_2c_3l_2e_4h_1h_2g_1l_4)$$
$$(d_1f_2e_3k_3k_4g_2f_4d_4i_4j_2i_3)(a)(f_3)(h_4)(k_1)(l_3).$$

The relation (E'') written in terms of S and T is $[(TS)^3(TS^2)^3]^2 = 1$.

The group generated by two operators of orders two and three with product of order 11 *and commutator of order* 5 *which satisfy the further relation* $[(TS)^3(TS^2)^3]^2 = 1$ *is the simple group of order* 660.

URBANA, ILLINOIS,
OCTOBER, 1927.

On Triadic Cremona Nets of Plane Curves.

By Fay Farnum.

I. Introduction.

1. A Cremona transformation is fixed when its scheme of f-points and f-curves is known.

Those which have in each plane only one group of fundamental points (f-points), composed of points of equal multiplicity, are called symmetric and are well known. A correspondence is called semi-symmetric if, in each of the two planes, there are only two groups of f-points; of these the Jonquières case is a familiar example; all others have been discussed by Montesano (D).*

A correspondence is called triadic if, in each of the two planes, there are three and only three groups of f-points; such correspondences have received little attention; one isolated case was discussed by Montesano, (C)* and a more general set by Ruffini (A).*

It is the purpose of this paper to determine additional triadic cases; in it are developed:

(a) some triadic correspondences as products of previously existing correspondences,

(b) all triadic correspondences having simple fundamental points in both planes,

(c) some triadic correspondences having simple fundamental points in one plane only.

2. *Characteristic Square.* The characteristic square of the transformation is:

$$T \equiv \begin{matrix} n & r_1 & r_2 & \cdots & r_p \\ s_1 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1p} \\ s_2 & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_p & \alpha_{p1} & \alpha_{p2} & \cdots & \alpha_{pp} \end{matrix}$$

in which n is the order of the transformation; $r_1, \cdots, r_p$ the orders of the f-points in one plane; $s_1, \cdots, s_p$ the orders of the f-points in the other plane;

* The reference is explained in section 3.

and α_{ij} is the number of times the fundamental curve (f-curve) of s_i goes through r_j, or also the number of times the f-curve of r_j goes through s_i.

The characteristic square may be simplified by indicating a group of α_i f-points of multiplicity r_i by the symbol $[\alpha_i, r_i]$ and a group of β_k f-curves of order s_k by $[\beta_k, s_k]$. In designating the number of passages of a group of curves through that group of points, such that each curve passes through all but one of the points L times, and through the remaining one $L \pm 1$ times, use is made of the symbol L followed by a plus sign or a minus sign, according as it passes through the remaining point $L+1$ or $L-1$ times. The simplified characteristic square for the isologous net is

n	$[1, n-1]$	$[2(n-1), 1]$
$[1, n-1]$	$n-2$	1 .
$[2(n-1), 1]$	1	$0+$.

For a net of order n which has α_1 f-points of multiplicity $r_1, \cdots, \alpha_m$ of multiplicity r_m, the characteristic group of the transformation is given by

$$G_n \equiv [\alpha_1, r_1] \cdots [\alpha_m, r_m].$$

The values in T are subject to the following relations

(a) n, r_i, s_k are positive integers,

(b) α_{ij} are positive integers or 0,

(1) (c) $\sum \alpha_i r_i^2 = n^2 - 1$,

(2) $\sum \alpha_i r_i = 3(n-1)$.

These are called the equations of condition.

(d) Each f-curve is rational and is completely determined by its passages through the f-points.

(e) The f-curves intersect only in f-points.

(f) The total number of passages of f-curves through an i-fold f-point is $3i-1$.

3. *References.* The results necessary for this paper were obtained from the following articles, and will be referred to by the indicated letters:

(A) F. P. Ruffini: "Sullo risoluzione delle trasformazioni Cremoniane," p. 499, *Memoire Istituto Bologna,* Ser. 3, Vol. 8 (1877).

(B) D. Montesano: "Sur le retiomaloidiche di curve," *Rendiconti Accademia Napoli,* 1905.

(C) D. Montesano: "Sur i quadri caratteristici delle corrispondenze birationali piane," *Rendiconti Accademia Napoli,* Ser. 3, Vol. 21 (1915).

(D) D. Montesano: "Le corrispondenze birazionali piane emisimmetriche," *Rendiconti Accademia Napoli,* Ser. 3, Vol. 24 (1918).

II. Correspondences Developed by Products.

4. Products of symmetric and Ruffini correspondences will be shown to give the following:

(a) $G_n \equiv [3, n/2],\quad [4, (n-2)/4],\quad [(n-2)/4, 2] \qquad n \equiv 2 \pmod 4,$

(b) $G_n \equiv [6, 2n/5],\quad [4, (n-5)/10],\quad [(n-5)/10, 2] \qquad n \equiv 5 \pmod{10},$

(c) $G_n \equiv [7, 3n/8],\quad [4, (n-8)/16],\quad [(n-8)/16, 2] \qquad n \equiv 8 \pmod{16},$

(d) $G_n \equiv [8, 3n/17],\quad [4, (n-17)/34],\quad [(n-17)/34, 2] \qquad n \equiv 17 \pmod{34}.$

(a) For any positive integral value of v there exists a Ruffini group $G_{2v+1} \equiv [4, v],\ [v, 2]$, which is geometric. Form the product of this group and the quadratic transformation, that is,

$$G_{2v+1} \equiv [3, 0],\ [4, v],\ [v, 2],$$
$$G_2 \equiv [3, 1].$$

The latter is so chosen that none of the f-points of the two fundamental systems are common as indicated by

$2v+1$	0	0	0	$[4, v]$	$[v, 2]$
2	1	1	1		
1	0	1	1		
1	1	0	1		
1	1	1	0		

This product gives

$$G_n \equiv [3, r],\ [4, v],\ [v, 2],$$

where

$$n = 2(2v+1),\quad r = 2v+1;$$

hence we have

$$G_{4v+2} \equiv [3, 2v+1],\ [4, v],\ [v, 2], \qquad (v = 1, 2, 3, \cdots)$$

or, in terms of n,

$$G_n \equiv [3, n/2],\ [4, (n-2)/4],\ [(n-2)/4, 2] \qquad n \equiv 2 \pmod 4.$$

For $n = 6$, $\quad G_6 \equiv [3, 3],\ [4, 1],\ [1, 2],$

$n = 10$, $\quad G_{10} \equiv [3, 5],\ [4, 2],\ [2, 2]$ or $G_{10} \equiv [3, 5],\ [6, 2].$

The latter degenerates into a semi-symmetric case, since $(n-2)/4=2$ for $n\equiv 10$, but this is true for no other value of n.

Since there are in this Cremona group $3P_{n/2}$, there will be 3 fundamental straight lines in π, hence in the fundamental system of the other plane, or in the conjugate group, there will be 3 simple f-points.

Let the multiplicities of the other two point groups of the conjugate group be x and y. They must satisfy (1) and (2) hence

$$x=n/2-1, \qquad y=4,$$

and the conjugate characteristic group is

$$G_n\equiv[4, n/2-1],\ [(n-2)/4, 4],\ [3, 1].$$

Since the fundamental straight lines are determined by two basis points of multiplicity $n/2$ and meet the fundamental system in no other points, the characteristic square may be indicated by

n	$[3, n/2]$,	$[4, (n-2)/4]$,	$[(n-2)/4, 2]$
$[4, n/2-1]$	a	$b\pm$	c
$[(n-2)/4, 4]$	d	e	$f\pm$
$[3, 1]$	$1-$	0	0

and, according to principles previously stated, the following equations must be satisfied:

$$n^2/2=4a(n/2-1)+4d(n-2)/4+2$$
$$n(n-2)/4=3b(n/2-1)+(b\pm 1)(n/2-1)+4e(n-2)/4$$
$$2n=4c(n/2-1)+4[(n-2)/4-1]f+4(f\pm 1)$$
$$n(n-2)/8=3ab+a(b\pm 1)+(n-2)de/4$$
$$n=4ac+[(n-2)/4-1]df+(f\pm 1)d$$
$$(n-2)/2=3bc+(b\pm 1)c+[(n-2)/4-1]ef+e(f\pm 1).$$

The solutions of these equations give the following values for the characteristic squares:

If v is odd, let $v=2\alpha+1$ and $n=4v+2=8\alpha+6$ with $n\equiv 6 \pmod 8$, we have

n	$[3, n/2]$	$[4, (n-2)/4]$	$[(n-2)/4, 2]$
$[4, n/2-1]$	$(n-2)/4$	$(n-6)/8+$	1
$[(n-2)/4, 4]$	2	1	$0+$
$[3, 1]$	$1-$	0	0

or

$8\alpha+6$	$[3, 4\alpha+3]$	$[4, 2\alpha+1]$	$[2\alpha+1, 2]$
$[4, 4\alpha+2]$	$2\alpha+1$	$\alpha+$	1
$[2\alpha+1, 4]$	2	1	$0+$
$[3, 1]$	$1-$	0	0

If v is even, let $v=2\alpha$. Then $n=4v+2=8\alpha+2$, with $n\equiv 2 \pmod 8$ and we obtain

n	$[3, n/2]$	$[4, (n-2)/4]$	$[(n-2)/4, 2]$
$[4, n/2-1]$	$(n-2)/4$	$(n-2)/8-$	1
$[(n-2)/4, 4]$	2	1	$0+$
$[3, 1]$	$1-$	0	0

or

$8\alpha+2$	$[3, 4\alpha+1]$	$[4, 2\alpha]$	$[2\alpha, 2]$
$[4, 4\alpha]$	2α	$\alpha-$	1
$[2\alpha, 4]$	2	1	$0+$
$[3, 1]$	$1-$	0	0 .

(b) Similarly form the product of $G_{2v+1}\equiv[6, 0]\ [4, v]\ [v, 2]$ and $G_5\equiv[6, 2]$. We obtain the following correspondences:

For v odd, $n=20\alpha+15$ with $n\equiv 15 \pmod{20}$,

$20\alpha+15$	$[6, 8\alpha+6]$	$[4, 2\alpha+1]$	$[2\alpha+1, 2]$
$[4, 10\alpha+1]$	$4\alpha+2$	$\alpha+$	1
$[2\alpha+1, 10]$	4	1	$0+$
$[6, 2]$	$1-$	0	0 .

For v even, $n=20\alpha+5$ with $n\equiv 5 \pmod{20}$,

$20\alpha+5$	$[6, 8\alpha+2]$	$[4, 2\alpha]$	$[2\alpha, 2]$
$[4, 10\alpha]$	4α	$\alpha-$	1
$[2\alpha, 10]$	4	1	$0+$
$[6, 2]$	$1-$	0	0 .

(c) The product of $G_{2v+1}\equiv[7, 0]\ [4, v]\ [v, 2]$ and $G_8\equiv[7, 3]$ gives for v even, $n\equiv 8 \pmod{32}$, $n>40$ with

$32\alpha+8$	$[7, 12\alpha+3]$	$[4, 2\alpha]$	$[2\alpha, 2]$
$[4, 16\alpha]$	6α	$\alpha-$	1
$[2\alpha, 16]$	6	1	$0+$
$[7, 3]$	$1+$	0	0 ,

and for v odd, $n \equiv 24 \pmod{32}$,

$$\begin{array}{cccc} 32\alpha+24 & [7, 12\alpha+9] & [4, 2\alpha+1] & [2\alpha+1, 2] \\ [4, 16\alpha+8] & 6\alpha+3 & \alpha+ & 1 \\ [2\alpha+1, 16] & 6 & 1 & 0+ \\ [7, 3] & 1+ & 0 & 0 \ . \end{array}$$

(d) The product of $G_{2v+1} \equiv [8, 0]\ [4, v]\ [v, 2]$ and $G_{17} \equiv [8, 6]$ gives, for v odd, $n \equiv 51 \pmod{68}$ with

$$\begin{array}{cccc} 68\alpha+51 & [8, 24\alpha+18] & [4, 2\alpha+1] & [2\alpha+1, 2] \\ [4, 34\alpha+17] & 12\alpha+6 & \alpha+ & 1 \\ [2\alpha+1, 34] & 12 & 1 & 0+ \\ [8, 6] & 2+ & 0 & 0 \ , \end{array}$$

and, for v even, $n \equiv 17 \pmod{68}$ with

$$\begin{array}{cccc} 68\alpha+17 & [8, 24\alpha+6] & [4, 2\alpha] & [2\alpha, 2] \\ [4, 34\alpha] & 12\alpha & \alpha- & 1 \\ [2\alpha, 34] & 12 & 1 & 0+ \\ [8, 6] & 2+ & 0 & 0 \ . \end{array}$$

5. *Product of two Ruffini correspondences.* The product of two Ruffini ($G_{2v+1} \equiv [4, v]\ [v, 2]$) groups of the same order, made in such a way that in the intermediate plane, the two fundamental systems have the double points in common, gives

$$G_{4v^2+1} \equiv [4, 2v^2 - v]\ [v, 4v]\ [4, v].$$

Use is made of the theorem (D, p. 18): "Given two plane birational correspondences $K_{\pi\pi'}$, $K_{\pi'\pi''}$ represented respectively by the squares

$$Q \equiv \begin{array}{|ccc|} \hline n & [\alpha, r'] & \cdots \\ [\alpha, r] & w \pm & \cdots \\ \vdots & \vdots & \\ \hline \end{array}\ \begin{matrix} \pi' \\ \\ \\ \pi \end{matrix} \qquad Q' \equiv \begin{array}{|ccc|} \hline n' & [\alpha, r_1'] & \cdots \\ [\alpha, r_1'] & w \pm & \cdots \\ \vdots & \vdots & \\ \hline \end{array}\ \begin{matrix} \pi'' \\ \\ \\ \pi' \end{matrix}.$$

If in the plane π' the α multiple points of order r' coincide with the α multiple points of order r_1', the resulting correspondence $K_{\pi\pi''}$ is given by

$$Q'' \equiv \begin{array}{|ccc|} \hline n'' & [\alpha, m_0''] & \cdots \\ [\alpha, m_0] & w'' \pm & \cdots \\ \vdots & \vdots & \\ \hline \end{array}\ \begin{matrix} \pi'' \\ \\ \\ \pi \end{matrix}$$

in which the term w'' has the sign + or — according as the signs of w, w'

are identical or opposite. Hence we obtain for v odd,

$4v^2+1$	$[4, 2v^2-v]$	$[v, 4v]$	$[4, v]$
$[4, 2v^2-v]$	v^2-v	$2v-1$	$(v-1)/2$
$[v, 4v]$	$2v-1$	$4-$	1
$[4. v]$	$(v-1)/2$	1	0 ,

for v even,

$4v^2+1$	$[4, 2v^2-v]$	$[v, 4v]$	$[4, v]$
$[4, 2v^2-v]$	v^2-v	$2v-1$	$v/2$
$[v, 4v]$	$2v-1$	$4-$	1
$[4, v]$	$v/2-$	1	0 .

III. Triadic Correspondences with Simple Points in Both Planes.

6. *Classification.* Let there be given between two planes, π and π', a birational correspondence of order n with simple f-points in both planes. The number, β, of simple f-points in π equals the number of fundamental straight lines in π'. In general each fundamental straight line is determined by two f-points, of multiplicities whose sum is n, as p and $n-p$. There are two cases, according as

(a) $p < n/2$ (b) $p = n/2$ (n even).

(a) $p < n/2$. The value of p must be the same for all the fundamental straight lines in π', otherwise there would be more than one point $n-p$, where $n-p > n/2$, which is impossible. Hence there must be, in π, β points of multiplicity p and one of multiplicity $n-p$.

(b) $p = n/2$. In this case $\alpha_{n/2} = 1, 2, 3$, since a homaloidal net cannot have more than 3 basis points of multiplicity $n/2$ (B, p. 11).

(1) In π', $\alpha_{n/2} = 1$; then other f-points of π' must have multiplicity less than $n/2$, hence there is no pair of f-points whose sum is n, therefore no fundamental straight lines. Consequently in π there will be no simple f-points; so this case is excluded.

(2) In π', $\alpha_{n/2} = 2$; then in π' there will be one fundamental straight line, hence one simple f-point in π.

(3) In π', $\alpha_{n/2} = 3$; then in π' there will be three fundamental straight lines, consequently three simple f-points in π.

From these results we have the following theorem: *If between two planes, π and π', there exists a birational correspondence of order n, and if in π there are β simple f-points, then in π' there are*

(1) *β f-points of multiplicity p and one f-point of multiplicity $n-p$, or*

(2) *if $\beta = 1, 3$ there may be*
for $\beta = 1$, two f-points of multiplicity $n/2$,
for $\beta = 3$, three f-points of multiplicity $n/2$.

We will discuss the cases:

I. $\beta =$ any positive integer, $p < n/2$,

II. $\beta = 1, 3$, $p = n/2$ (n even).

7. Case I. $\beta =$ any positive integer, $p < n/2$. The correspondence may be designated by

in π $\quad G_n \equiv [1, m]\ [\alpha, s]\ [\beta, 1]$,

in π' $\quad G_n \equiv [1, n-p]\ [\beta, p]\ [\alpha, 1]$, $\qquad m + s = n$.

The case of $p = 1$ will not be considered since there the correspondence becomes a Jonquières, semi-symmetric correspondence. If p and n are used as parameters, the only set of solutions satisfying the equations of condition is

$$s = (n-1)/p, \qquad m = n - (n-1)/p,$$
$$\beta = 2(n-1)/p - 1, \qquad \alpha = 2p - 1.$$

These values will be positive integers if and only if $n \equiv 1 \pmod{p}$ and $n > 1$. By methods previously used, we obtain the following as the characteristic square of the correspondence. $k = (n-1)/p$.

$kp+1$	$[1, kp-p+1]$	$[2k-1, p]$	$[2p-1, 1]$
$[1, kp-k+1]$	$kp+2-p-k$	$p-1$	1
$[2p-1, k]$	$k-1$	1	$0+$
$[2k-1, 1]$	1	$0+$	0 .

The correspondence has so far only been shown to satisfy the equations of condition; it yet remains to prove it geometric. A correspondence is geometric if it is the product of geometric correspondences. The series of products which result in a correspondence is called its series of origin. The reverse of the series of origin of the correspondence here considered is a series of products of the original by the quadratic correspondence, formed so that the three simple f-points of G_2 coincide with the three f-points of highest multiplicity in the multiplicand, and is as follows:

$$G_n \equiv [1, n-p] \quad [2(n-1)/p-1, p] \quad [2p-1, 1],$$
$$G_{n-p} \equiv [1, n-2p] \quad [2(n-1)/p-3, p] \quad [2p-1, 1],$$
$$G_{n-2p} \equiv [1, n-3p] \quad [2(n-1)/p-5, p] \quad [2p-1, 1],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$G_{n-ip} \equiv [1, n-(i+1)p] \quad [2(n-1)/p-(2i+1), p] \quad [2p-1, 1];$$

and for $i = (n-1)/p-1$, we have

$$G_{p+1} \equiv [1, n-(n-1)p] \quad [1, p] \quad [2p-1, 1].$$

The last correspondence reduces to

$$G_{p+1} \equiv [1, p] \quad [2p, 1],$$

which is the well-known geometric isologous correspondence. Hence G_n is a geometric correspondence.

If $p = 2$, the result is the general Ruffini correspondence (D, p. 499).

That the above correspondence be self-conjugate, the necessary and sufficient condition is that $k = p$.

8. Case II. $\beta = 1, 3 \quad p = n/2$.

There are the following possible cases:

(1) $G_n \equiv [1, n-q] \quad [\alpha, q] \quad [3, 1],$ $\qquad q < n/2$
$\quad\; G_n \equiv [3, n/2] \quad [1, s] \quad [\alpha, 1],$

(2) $G_n \equiv [3, n/2] \quad [\alpha, s] \quad [3, 1],$
$\quad\; G_n \equiv [3, n/2] \quad [\alpha, p] \quad [3, 1],$

(3) $G_n \equiv [3, n/2] \quad [2, r] \quad [1, 1],$
$\quad\; G_n \equiv [2, n/2] \quad [1, s] \quad [3, 1],$

(4) $G_n \equiv [2, n/2] \quad [1, m] \quad [1, 1],$
self-conjugate,

(5) $G_n \equiv [2, n/2] \quad [1, m] \quad [\alpha, 1],$
$\quad\; G_n \equiv [1, n-p] \quad [\alpha, p] \quad [1, 1].$

The equations of condition determine the unknown values with the following results:

(1) No triadic correspondence except the Ruffini case,

(2) The correspondence is given by

n	$[3, n/2]$	$[9-72/(n+4), n+4/6]$	$[3, 1]$
$[3, n/2]$	$n/4$	$(n+4)/12$	$1-$
$[9-72/(n+4), n+4/6]$	$(n+4)/12$	$z \pm$	0
$[3, 1]$	$1-$	0	0

where $z = (n+4)(n/4 \pm 1)/9(n-4)$, $n = 68, 32, 20, 8$ only,

(3) No triadic case,

(4) $\alpha = 8 + 32/(n+2)$, hence $n = 2, 6, 14, 30$, giving correspondences for $n = 6, 14, 30$ only,

(5) No solution.

IV. Some Triadic Birational Correspondences with Simple Fundamental Points in One Plane Only.

9. *Classification.* Let $\beta =$ number of simple f-points in plane π, then, by the theorem previously discussed, there will be in π',

I. For β any integer, β f-points of multiplicity p $(p < n/2)$ and one f-point of multiplicity $n-p$, or

II. For $\beta = 1$ and n even, two f-points of multiplicity $n/2$, or

III. For $\beta = 3$ and n even, three f-points of multiplicity $n/2$.

10. Case I. The correspondence may be represented by

n	$[1, v]$	$[\alpha, s]$	$[\beta, 1]$	
$[1, n-p]$	a	b	1	
$[\beta, p]$	c	d	$0+$	
$[\alpha, t]$	e	$f \pm$	0	$t \neq 1$.

The cases for which $p = 1$ and $p = 2$ give the semi-symmetric Jonquières and general Ruffini triadic correspondences, respectively, hence, will not be discussed.

From the characteristics stated in § 2 we obtain the following equations:

by (d) (3) $c^2 + \alpha d^2 = p^2$,
(4) $(n-p)^2 + 1 = a^2 + \alpha b^2 + \beta$,
(5) $t^2 = e^2 + \alpha f^2 \pm 2f$,

by (f) (6) $e + \alpha f \pm 1 = 3t - 1$,
(7) $c + \alpha d = 3p + 2$,
(8) $3(n-p) - 1 = a + \alpha b + \beta$,

by (e) (9) $ca + \alpha db + 1 = p(n-p)$.

Let $p=3$, then $c<p$, hence $c=0$, 1 or 2; similarly $d<p$, hence $d=0$, 1, or 2. The equations (3) and (7) are satisfied by $c=2$, $\alpha=5$, and $d=1$, giving $b=(n-2)/3$ or $(n-4)/3$. In equations (5) and (6) we must have

$$t<p,\ \therefore\ t=2;\quad e<t,\ \therefore\ e=0 \text{ or } 1;\quad f<t,\ \therefore\ f=0 \text{ or } 1.$$

Under these conditions the only solution is with a minus sign for f, and is given by $t=2$, $e=1$, $f=1$. By methods previously used we complete the following correspondence, where the parameter is $k=(n-2)/3$:

$3k+2$	$[1,2k]$	$[5,k+1]$	$[2(k-1),1]$	
$[1,3k-2]$	$2(k-1)$	k	1	
$[2(k-1),3]$	2	1	$0+$	
$[5,2]$	1	$1-$	0	$(k=2,3,4\cdots)$.

The series of origin is given by

$$\begin{aligned} G_{k+1} &\equiv [1,k]\ [2k,1], \\ G_{2k+2} &\equiv [1,k]\ [3,k+1]\ [2k,1], \\ G_{3k+2} &\equiv [1,2k]\ [2,k+1]\ [3,k+1]\ [2(k-1),1]. \end{aligned}$$

Similarly for $p=4$, we obtain the following correspondences:

$4k+2$	$[1,3k]$	$[7,k+1]$	$[2(k-2),1]$	$k=(n-2)/4$
$[1,4k-2]$	$3(k-1)$	k	1	
$[2(k-2),4]$	3	1	$0+$	
$[7,3]$	2	$1-$	0	,

and

$4k+1$	$[1,3k+1]$	$[7,k]$	$[2k-1,1]$	$k=(n-1)/4$
$[1,4k-2]$	$3(k-1)$	k	1	
$[2(k-2),4]$	3	1	$0+$	
$[7,3]$	2	$1-$	0	.

11. *General Correspondence.* The correspondences for $p=3$, 4 suggest the generalized case $p=p$ and the following analogous correspondence:

$kp+2$	$[1,k(p-1)]$	$[2p-1,k+1]$	$[2(k+2-p),1]$	
$[1,kp+2-p]$	$(p-1)(k-1)$	k	1	
$[2(k-p+2),p]$	$p-1$	1	$0+$	
$[2p-1,p-1]$	$p-2$	$1-$	0	$k=(n-2)/p$.

It is necessary that $k+2-p>0$, hence we may have $p=3,\ 4,\ \cdots$, and

$k = 2, 3, \cdots$ This correspondence can be shown to check in the necessary equations and the following gives the series of reductions by quadratic inversions to the Jonquières correspondence:

$$G_{kp+2} \equiv [1, k(p-1)]\ [2p-1, k+1]\ [2(k-p+2), 1],$$
$$G_{k(p-1)+2} \equiv [1, k(p-2)]\ [2p-3, k+1]\ [2(k-p+3), 1],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$G_{k(p-i)+2} \equiv [1, k\{p-(i+1)\}]\ [2p-(2i+1), k+1]\ [2\{k-p+(i+2)\}, 1].$$

The last gives for $i = p-2$

$$G_{2k+2} \equiv [1, k]\ [3, k+1]\ [2k, 1],$$

from which a quadratic inversion gives

$$G_{k+1} \equiv [1, k]\ [2k, 1].$$

12. Case II. $\beta = 1$, n even, two f-points of multiplicity $n/2$.

The possible solutions are indicated by:

A.	n	$[2, n/2]$	$[\alpha, s]$	$[1, t]$	*B.*	n	$[2, n/2]$	$[2, s]$	$[1, t]$
	$[2, p]$	$a+$	b	c		$[2, p]$	a	$b \pm$	c
	$[\alpha, u]$	d	$e \pm$	f		$[2, u]$	$d+$	e	f
	$[1, 1]$	1	0	0		$[1, 1]$	1	0	0 .

From the principles stated in § 2 we have for *A*,

$$t^2 + 1 = 2c^2 + \alpha f^2 \tag{10}$$

$$3t - 1 = 2c + \alpha f. \tag{11}$$

The solving of (10) and (11) for t in terms of α and c gives

$$t = \{-3(2c+1) \pm \alpha^{1/2}[2(\alpha-7)c^2 + 4c + 10 - \alpha]^{1/2}\}/(\alpha-9), \qquad \alpha \neq 9. \tag{12}$$

In order that t be rational it is necessary that

$$2(7-\alpha)c^2 - 4c + \alpha - 10 \leq 0$$

which gives for $c = 0$, $\alpha \leq 10$

$$c \geq 1, \quad \alpha \geqq 7 - (4c+3)/(2c^2-1).$$

The latter relation shows that, for $\alpha < 7$, it is necessary to consider only those values of c which make $(4c+3)(2c^2-1) > 1$, that is, $c = 1, 2$. Hence the following cases will be discussed:

(a) $c = 0$, (b) $c = 1$, (c) $c = 2$, (d) $\alpha \geq 7$, $\alpha = 7, 8, 9, 10 \cdots$.

(a) $c=0$, $\alpha \leqq 10$, $\alpha \neq 9$. By the equations of condition, we find that the only new triadic correspondence is a G_{10} having in π the characteristic group $[2,5]$ $[5,3]$ $[1,2]$ and in π', $[2,3]$ $[5,4]$ $[1,1]$.

(b) $c=1$, $\alpha \neq 9$. Equation (12) gives $t=1$, or $-1+18/(9-\alpha)$; hence the only solutions are given by

$\alpha=3$	6	7	8
$t=2$	5	8	17
$n=4$	$\begin{cases}10\\22\end{cases}$	$\begin{cases}16\\72\end{cases}$	34.

All of these give geometric correspondences.

(c) $c=2$. For the given equations, the only rational values for t are 1 and 8. The former has been discussed and the latter gives $r=19/3$, which is excluded. Hence this case adds no triadic correspondence.

(d) $\alpha \geqq 7$. If $\alpha=7$, equations (10) and (11) give

$$t=\{3(2c+1) \pm [7(4c+3]^{\frac{1}{2}}\}/2.$$

In order that t be rational it is necessary that $c \equiv 1 \pmod 7$, that is, $c=7x+1$; it is also necessary that $7(4c+3)=7(28x+7)=49(4x+1)$ be a square. The resulting values are

$t=21k^2+28k+8$	and	$t=21k^2+14k+1$
$f=\ 7k^2+10k+3$		$f=\ 7k^2+\ 4k$
$c=\ 7k^2+\ 7k+1$		$c=\ 7k^2+\ 7k+1$
$r=\ 7$		$r=\ 7.$

The first gives $t=1$, and the second a mixed number for s, hence we have no new correspondences.

$\alpha=8$. Similarly it can be shown that this case gives no new correspondences.

$\alpha=9$. From (12) we find that for c to be rational, it is necessary that

$$4(3t-1)^2+(8+6t)$$

be a square, which gives 1 as the only integral value of t.

$\alpha=10$. The only value possible for t is 1, hence no new triadic correspondence.

Further values of α ($\alpha > 10$) could be similarly investigated.

B. The only possible value for t is 1.

13. Case III. $\beta = 3$, n even, 3 f-points of multiplicity $n/2$. If we let $s = 2$, $b = 0$, we obtain as one correspondence for this case,

$$G_n \equiv [3, n/2]\ [4, (n-2)/4]\ [(n-2)/4, 2],$$

which is the same correspondence that was previously developed as a product of the Ruffini and quadratic correspondences.

14. *Conclusion.* This completes the discussion of all triadic correspondences involving simple points in both planes, hence of all self-conjugate triadic correspondences with simple points. It also gives some solutions for each of the cases of triadic correspondences involving simple points in one plane only.

Number Relations between Types of Extremals Joining a Pair of Points.*

By D. E. Richmond.†

1. *Introduction.* Writers on the calculus of variations have heretofore been primarily concerned with minimizing or maximizing a given integral. Accordngly, attention has been centered on extremal arcs which have at most one conjugate point to a given point. The Euler equation, however, gives merely the condition that the integral be stationary. A new field has been opened up by H. M. Morse ‡ in a paper which considers extremal arcs bearing an arbitrary number of conjugate points. In this paper, Morse develops a parallelism between types of critical points and types of extremal arcs, in which the type of an extremal arc, whose end-points are not conjugate to each other, is determined by the number of points within the arc conjugate to the initial point. Hence, a given extremal arc has a type number equal to the number of such conjugate points.

The primary object of the present paper is to establish the completeness of a set of relations obtained by Morse, between the numbers of extremals of different type, joining two fixed points in an ordinary calculus of variations problem. Morse's relations are obtained for the parametric case under certain general boundary conditions and under the hypothesis of the existence of a field of extremals covering the region of operation.

By the aid of the field of extremals which are taken as the curves $x =$ constant, Morse first reduces the problem to one in non-parametric form. When Morse does not wish to exclude the case where the end-points of an extremal arc are mutually conjugate, he assumes that the integrand is analytic. We shall exclude this possibility and accordingly require less in the way of hypothesis. Reference to Morse's paper will show that the relations in question hold for the following non-parametric problem. Given the integral

$$(1) \qquad \int_{x_0}^{x_1} f(x, y, y')\, dx,$$

* Presented to the Society, May 7, 1927.

† National Research Fellow in Mathematics.

‡ To appear in the June issue of the *Transactions of the American Mathematical Society.*

where $f(x, y, y')$ is of class C'''' for any number y' and for (x, y) in an open region R of the (x, y) plane including the x-axis. In R let there be given two curves

$$\gamma_1 \quad y = y_1(x), \quad -\infty < x < \infty,$$
$$\gamma_2 \quad y = y_2(x), \quad y_1(x) < y_2(x),$$

in which $y_1(x)$ and $y_2(x)$ are analytic in x.

I. We suppose that the curves γ_1 and γ_2 are extremal-convex relative to the region R'

$$y_1(x) \leqq y \leqq y_2(x), \quad -\infty < x < \infty,$$

in the sense that extremals tangent to γ_1 and γ_2 lie outside of R' in the neighborhood of the point of contact.

II. We suppose that there exists a positive constant e so small that two points in R' on any two lines $x = k$ and $x = k + e$, respectively, can be joined by a unique extremal arc of (1), upon which there exists no pair of conjugate points.

That (I) and (II) are in fact satisfied in the examples we consider, will appear in Lemmas 2 and 4.

Let A and B be any two points in R' not on the same line $x =$ constant. It follows from I and II that there exists at least one extremal E joining A to B in R'. We suppose that on no extremal E is B conjugate to A. Let n be the maximum type of any extremal joining A to B. Let M_i $(i = 0, 1, \cdots, n)$ be the number of extremals E of type i. Then the following theorem holds.

THEOREM. *Between the numbers M_i corresponding to the set of extremals joining any two fixed points in R' there exist the following relations:*

$$\begin{aligned}
&M_0 \geqq 1, \\
&M_0 - M_1 \leqq 1, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&(-1)^{n-1} [M_0 - M_1 + \cdots (-1)^{n-1} M_{n-1}] \geqq 1, \\
&M_0 - M_1 + \cdots (-1)^n M_n = 1.
\end{aligned} \tag{2}$$

Morse has given a proof, not yet published, that if $n > 0$, $M_0 \geqq 2$. No use will be made of this fact, other than to restrict the relations whose completeness is to be proved. No further restrictive relations are known. Let (2′) denote the set of relations (2), in which the first inequality is replaced by $M_0 \geqq 2$.

An obvious method of proving the absence of new independent relations is to show that, given an arbitrary set of positive integers M_i $(i = 0, 1, \cdots, n)$ satisfying (2′), there exists a calculus of variations problem of the form (1) such that the number of extremal arcs of type i joining a given pair of points is equal to M_i. The present paper will give such an existence proof for $n = 2$. Work is in progress on the relations involving extremals of higher type.

It will be well to point out that among the relations (2) or (2′), the next to the last may be omitted. For example, if $n = 2$, $M_0 - M_1 \leqq 1$ follows from $M_0 - M_1 + M_2 = 1$.

We shall show that there exists an example corresponding to each set of positive integers M_0, M_1, and M_2 satisfying

$$M_0 \geqq 2, \qquad M_0 - M_1 + M_2 = 1. \tag{3}$$

2. *The integral.* All of our examples will be obtained by an appropriate choice of $\phi(y)$ and b in the integral

$$\int_0^b [y'^2 - \phi(y)]\, dx. \tag{4}$$

In every case, we require that:

(A) *The function $\phi(y)$ be of class C'''' in an interval*

$$(I) \quad -h' - e' \leqq y \leqq h + e,$$

where h, h', e, and e' are positive constants to be more explicitly chosen later;

(B) $\phi(y)$ *have a minimum at* $y = 0$,

maxima at $y = h$ *and* $y = -h'$, *and* $\phi'(y) \neq 0$ *elsewhere in* (I);

(C) $\phi(0) = 0$,

$\phi(y) > 0, \quad y \neq 0$;

(D) $\phi(-y) = \phi(y), \quad 0 \leqq y \leqq \gamma, \quad \gamma \leqq h$ *and* $\leqq h'$, *where γ is a positive constant.*

3. *The extremals.* We shall describe the extremals of (4), making use of the general properties of $\phi(y)$ given in § 2.

The Euler equation corresponding to (4) is

$$y'' = -\phi'(y)/2. \tag{5}$$

Let us first consider the extremals only insofar as they remain in the strip S

$$(S) \quad -h' \leqq y \leqq h, \quad -\infty < x < \infty.$$

In the first place, $y = h$ and $y = -h'$ are extremals. The only other extremal of the form $y =$ constant, is $y = 0$. By the existence theorem for differential equations, there exist no other solutions of (5) tangent to $y = h$, $y = -h'$, or $y = 0$.

Because of the property (B) of $\phi(y)$, *any extremal, other than* $y = 0$, *which has points within* S, *must be concave toward the* x*-axis and hence must intersect it at least once.*

Since x does not occur explicitly in (5), it is clear that if $y = g\ (x)$ is an extremal in S, so also is $y = g\ (x + \alpha)$, where α is an arbitrary constant. It will suffice for our purposes to consider extremals which pass through the origin.

Now (5) admits the first integral

$$y'^2 + \phi(y) = c^2. \tag{6}$$

Consider the extremal E through the origin with the initial slope y_0'. Since $\phi(0) = 0$, it follows from (6) that $c^2 = y_0'^2$ so that c in (6) may be interpreted as the initial slope. Let E, the extremal through $(0, 0)$ with initial slope c, be represented by $y = g\ (x, c)$.

For $c = 0$, we obtain simply the x axis. For $c > 0$, the discussion will be divided into three cases, according as $c^2 < \phi(h)$, $c^2 = \phi(h)$, or $c^2 > \phi(h)$.

Case I. $c^2 < \phi(h)$. From (6) and (5) it follows that as the ordinate y on E increases, y' decreases until for some constant $\bar{y} < h$, $\phi(\bar{y}) = c^2$. At the point $P = (\bar{x}, \bar{y})$ on E,

$$y' = g_x(\bar{x}, c) = 0.$$

Since E is concave toward the x axis, it has a maximum at P. From (6), the slope of E is numerically the same at all points with the same ordinate, and E is symmetric with respect to P. Hence, for $c^2 < \phi(h)$, the extremal $y = g\ (x, c)$ intersects the x axis again at the point $(2\bar{x}, 0)$.

As c^2 increases from 0 to $\phi(h)$, the maximum ordinate $\bar{y}$ of E, given by $\phi(\bar{y}) = c^2$, increases from 0 to h, since

$$\phi'(\bar{y}) > 0, \quad 0 < \bar{y} < h.$$

Case II. $c^2 = \phi(h)$. Here (6) becomes

$$y'^2 = \phi(h) - \phi(y).$$

Then $y' \neq 0$ for $0 \leqq y < h$, so that y increases steadily on E. Clearly E cannot have a maximum for $y < h$. Now $y' = 0$ for $y = h$, but E is not tangent to $y = h$ because of the existence theorem. It follows that E approaches $y = h$ asymptotically. Obviously then, E does not intersect the x axis again.

Case III. $c^2 > \phi(h)$. The corresponding extremals through $(0, 0)$ pass out of S, intersecting $y = h$. After leaving the origin, they do not intersect each other or the extremal considered in Case II. Let us follow these extremals into the strip,

$$h < y \leqq h + e, \quad -\infty < x < \infty.$$

In this e strip, the extremals are concave away from the x axis, since $\phi'(y) < 0$. Hence they do not return to S before intersecting $y = h + e$. Therefore, in Case III, as in Case II, the extremals through the origin do not intersect the x axis again.

If we now consider negative values of c, we can make a similar classification of extremals. Extremals through the origin intersect the x axis again, if $c^2 < \phi(-h')$, otherwise not.

We may summarize our results in the following lemma.

LEMMA 1. *Given the integral*

$$\int_0^b [y'^2 - \phi(y)]\, dx,$$

where $\phi(y)$ has the properties prescribed in § 2. An extremal which passes through the origin with the slope $c \neq 0$ intersects the x axis again if

$$c > 0 \quad \textit{and} \quad c^2 < \phi(h),$$
$$\textit{or } c < 0 \quad \textit{and} \quad c^2 < \phi(-h'), \quad \textit{otherwise not.}$$

Let a region S' be defined by

$$(S') \quad -d' - h' \leqq y \leqq h + d, \quad -\infty < x < \infty, \quad 0 < d < e, \; 0 < d' < e'.$$

Then S' is extremal-convex, for since $\phi'(h + d) < 0$, it follows from (5) that an extremal tangent to $y = h + d$ lies outside of S' in the neighborhood of the point of contact. A similar statement applies at the boundary, $y = -h' - d'$.

LEMMA 2. *The region S' is extremal-convex.*

4. *Intersection of extremals with x axis.* We shall now consider an extremal E through the origin, whose initial slope satisfies the inequality

$$c^2 < \phi(\gamma),$$

where γ is the constant of § 2 (D), such that

$$\phi(-y) = \phi(y), \quad 0 \leqq y \leqq \gamma.$$

Then on E, y is a periodic function of x. The first intersection of E with the positive x axis occurs at $x = 2\bar{x}$ (cf. § 3), or more explicitly $x = 2\bar{x}(c)$. For convenience in notation we shall hereafter let

$$\eta(c) = 2\bar{x}(c).$$

For $c^2 < \phi(\gamma)$, the intersections of E with the positive x axis occur at

$$x = m\eta(c), \tag{7}$$

m a positive integer. By integrating (6), we have for $c > 0$,

$$\eta(c) = 2\bar{x}(c) = 2\int_0^{\bar{y}(c)} \frac{dy}{[c^2 - \phi(y)]^{\frac{1}{2}}}, \quad 0 < c^2 < \phi(h). \tag{8}$$

Here $\bar{y}(c)$ is obtained by solving

$$\phi(\bar{y}) = c^2, \quad \phi'(\bar{y}) \neq 0, \quad 0 < \bar{y} < h.$$

Equation (8) is valid in the indicated c interval although for the application in (7) we require that $c^2 < \phi(\gamma) \leqq \phi(h)$. For negative values of c, η (c) is defined by a formula similar to (8).

Now the integral (8) is improper because the integrand becomes infinite for $y = \bar{y}$. But in the neighborhood of $y = \bar{y}$,

$$[c^2 - \phi(y)]^{\frac{1}{2}} = (\bar{y} - y)^{\frac{1}{2}}\psi(y), \quad \psi(\bar{y}) \neq 0,$$

where $\psi(y)$ is continuous. The convergence of (7) follows according to well-known rules.

5. *Type of extremal arc.* Let $y = g\ (x, c)$ represent the one parameter family of extremals through $(0, 0)$, and $y = g(x, c_0)$ $0 < c_0^2 < \phi(\gamma)$, a particular extremal E of this family. If for abbreviation we let

$$m\eta(c_0) = a,$$

then by (7) the extremal arc E_a

$$(E_a) \quad y = g(x, c_0), \quad 0 \leqq x \leqq a,$$

joins two points on the x axis. We wish to determine its type, i. e., the number of points on E_a conjugate to $(0, 0)$. This will require a study of the

Jacobi equation.* Now all extremals of (4), for which $-h' < y < h$, are represented by

$$y = g(x + \alpha, c).$$

The Jacobi equation for (4), set up for the particular extremal E, $y = g(x + \alpha_0, c_0)$, is

$$w'' + (\phi''/2) g(x + \alpha_0, c_0) w = 0. \tag{9}$$

It is easily seen † that for $c_0 \neq 0$,

$$w_1(x) = g_a(x + \alpha_0, c_0) \text{ and } w_2(x) = g_c(x + \alpha_0, c_0) \tag{10}$$

are linearly independent solutions of (9). But $w_1(x) = g_a(x + \alpha_0, c_0) = g_x(x + \alpha_0, c_0)$ is the slope of E. For E_a, an extremal arc through the origin, $\alpha_0 = 0$, and $c_0^2 < \phi(\gamma)$. If P, P', P'', $\cdots$ are successive maxima and minima on E_a, P' is the first conjugate point to P, P'' is the second, and so on.

Since $w_2(0) = 0$ and $w_2'(0) \neq 0$, the conjugate points to $A = (0, 0)$ on E_a are given by the zeros of $w_2(x)$. By Sturm's separation theorem, the roots of linearly independent solutions of (9) separate each other. Hence the first conjugate point A' to the origin A lies between P and P' on E_a, the second between P' and P'' and so on. On E_a, therefore, there exist not less than $(m - 1)$ nor more than m points conjugate to A. Hence the *type of* E_a *is either* $(m - 1)$ *or* m. The decision rests with the sign of $\eta'(c)$ as we proceed to show.

6. *Type of extremal arc (cont.).* Now $\eta(c)$ satisfies the following identity in c.

$$g(m\eta(c), c) \equiv 0, \quad 0 < c^2 < \phi(\gamma).$$

Differentiating with respect to c,

$$g_c(m\eta(c), c) + m\eta'(c) g_x(m\eta(c), c) \equiv 0.$$

On setting $c = c_0$ and $m\eta(c_0) = a$, we have by (10),

$$w_2(a) = -m\eta'(c_0)\, w_1(a).$$

In the following discussion, we shall drop the subscript on c_0. There are three cases, depending on the sign of $\eta'(c)$. For simplicity, we restrict ourselves to positive values of c.

Case I. If $\eta'(c) > 0$, $w_2(a)$ and $w_1(a)$ have opposite signs.

---

* Cf. Bolza, *Vorlesungen über Variationsrechnung* (1909), p. 60. This book will hereafter be referred to as B.

† Cf. B., p. 74.

(a) If m is even, $w_1(a)$ is positive. Then $w_2(a)$ is negative, which means that $w_1(x)$ vanishes an odd number of times between $x = 0$ and $x = a$, hence $(m-1)$ times.

(b) If m is odd, $w_1(a)$ is negative. Then $w_2(a)$ is positive which means that $w_2(x)$ vanishes an even number of times between $x = 0$ and $x = a$, hence $(m-1)$ times.

For Case I, therefore, the type is $(m-1)$.

Case II. If $\eta'(c) < 0$, it follows similarly that the type is m, whether m is odd or even.

Case III. If $\eta'(c) = 0$, $w_2(a) = 0$ and the point $(a, 0)$ is conjugate to $A = (0, 0)$ on E_a. In the examples which we shall give extremal arcs whose end-points are conjugate will not occur.

LEMMA 3. *Given the extremal arc E_a through the origin represented by*

$$(E_a) \quad y = g(x, c_0), \quad 0 \leqq x \leqq a,$$

where $a = m\eta(c_0)$.

Then $g(a, c_0) = 0$,

and E_a is of type $m-1$, *if* $\eta'(c_0) > 0$;
m, *if* $\eta'(c_0) < 0$.

7. *Extremals joining points on neighboring parallels.*

LEMMA 4. *There exists a positive constant e so small that two points in S' on any two lines $x = k$ and $x = k + e$ respectively can be joined by a unique extremal arc upon which there exist no pairs of mutually conjugate points.*

Since S' is an extremal-convex region (Lemma 2), any two points P and Q in S' may be joined by at least one extremal arc (§ 1).

Now since $\phi''(y)/2$ is continuous in $-h' - d' \leqq y \leqq h + d$, it is bounded in the same interval. Say $\phi''(y)/2 \leqq M^2$, $-h' - d' \leqq y \leqq h + d$. Comparing the Jacobi equation (9) with

$$z'' + M^2 z = 0,$$

it follows by Sturm's comparison theorem, that no two conjugate points are separated by an x interval less than $\pi/M = e$, say. Hence, if P and Q are on the lines $x = k$ and $x = k + e$, respectively, there exists no pair of conjugate points on any arc PQ.

It now follows that the extremal arcs passing through P and intersecting $x = k + e$, form a field,* except for the point P itself. Hence through P and Q there can exist not more than one extremal arc. The lemma is therefore proved.

8. *Required properties of* $\eta(c)$. Let us turn from general considerations to the specific task of constructing examples in which M_0, M_1 and M_2 are arbitrary positive integers satisfying the relations.

$$M_0 \geqq 2, \qquad M_0 - M_1 + M_2 = 1. \tag{3}$$

We first note that all such integers are expressible in the form

$$\begin{aligned} M_0 &= 2 + p, & p &\geqq 0, \\ M_1 &= 2 + p + q, & q &\geqq 0, \\ M_2 &= 1 + q. \end{aligned} \tag{11}$$

We have four cases to consider:

Case I. q even, p even;
Case II. q odd, p even;
Case III. q even, p odd;
Case IV. q odd, p odd.

We shall choose for the points A and B to be joined by extremal arcs, the origin $(0, 0)$ and the point $(b, 0)$, respectively, in conformity with the limits of the integral (4).

We shall now describe the character of the function $\eta(c)$ to be used for a given p and q in (11).

Case I. q even, p even. Let $q = 2q_1$ and $p = 2p_1$. We shall so construct an example that:

(1) $b/3 < \eta(0) < b/2$, where $\eta(0) = \lim_{c \to 0} \eta(c)$;
(2) $\eta(c) = b/2$, $\eta'(c) > 0$, for $q_1 + 1$ values of $c > 0$;
(3) $\eta(c) = b/2$, $\eta'(c) < 0$, for q_1 values of $c > 0$;
(4) $\eta(c) = b$, $\eta'(c) > 0$, for $p_1 + 1$ values of $c > 0$;
(5) $\eta(c) = b$, $\eta'(c) < 0$, for p_1 values of $c > 0$;
(6) $\eta(c) > b/3$;
(7) $\eta'(c) \neq 0$ when $\eta(c) = b$ or $b/2$;
(8) $\eta(-c) = \eta(c)$.

* Cf. B., p. 307.

In (1), $\eta(0)$ gives the first conjugate point * to $(0, 0)$. According to (1) therefore, the extremal $y = 0$ is of type 2. Using this fact and Lemma 3, (where a is to be replaced by b) it follows that for an example for which $\eta(c)$ satisfies the conditions above

$$M_0 = 2(p_1 + 1) = 2 + p,$$
$$M_1 = 2p_1 + 2(q_1 + 1) = 2 + p + q,$$
$$M_2 = 1 + 2q_1 = 1 + q,$$

as required.

Case II. q odd, p even. Let $q = 2q_1 + 1$ and $p = 2p_1$. An example will be constructed so that:

(1) $b/2 < \eta(0) < b$;
(2) $\eta(c) = b/2$, $\eta'(c) > 0$, for $q_1 + 1$ values of $c > 0$;
(3) $\eta(c) = b/2$, $\eta'(c) < 0$, for $q_1 + 1$ values of $c > 0$;
(4) $\eta(c) = b$, $\eta'(c) > 0$, for $p_1 + 1$ values of $c > 0$;
(5) $\eta(c) = b$, $\eta'(c) < 0$, for p_1 values of $c > 0$;
(6) $\eta(c) > b/3$;
(7) $\eta'(c) \neq 0$ when $\eta(c) = b$ or $b/2$;
(8) $\eta(-c) = \eta(c)$.

It follows from (1) that the extremal $y = 0$ is of type 1. Using Lemma 3, it follows that for an example for which $\eta(c)$ satisfies the requirements above

$$M_0 = 2(p_1 + 1) = 2 + p,$$
$$M_1 = 1 + 2p_1 + 2(q_1 + 1) = 2 + p + q,$$
$$M_2 = 2(q_1 + 1) = 1 + q.$$

Case III. q even, p odd. Let $q = 2q_1$ and $p = 2p_1 + 1$. We seek an example for which

$$M_0 = 2 + 2p_1 + 1,$$
$$M_1 = 2 + 2p_1 + 2q_1 + 1,$$
$$M_2 = 1 + 2q_1.$$

Suppose all of the values of c at which the function $\eta(c)$ described in Case I takes on the values b and $b/2$ lie in the interval D,

$$(D) \quad -c_1 \leqq c \leqq c_1.$$

In the present case we start with a function $\eta(c)$ of the same character on D as the function $\eta(c)$ of Case I. Because of the nature of $\eta(c)$ on D

* Cf. B., p. 237.

we will obtain extremals whose type numbers sum up as in Case I. In addition we need in this case one extremal of type zero and one of type one. These we shall obtain by requiring that for two values of c we shall have respectively:

$$\eta(c) = b, \quad \eta'(c) > 0, \quad c > c_1;$$
$$\eta(c) = b, \quad \eta'(c) < 0, \quad c > c_1.$$

Case IV may now be treated in a manner similar to Case III, making use this time, however, of the results under Case II.

We have now described an $\eta(c)$ which is suitable for any example involving extremals of type 2 joining A to B, with no extremals of higher type. It is now necessary to so choose $\phi(y)$ that $\eta(c)$ has the character thus described.

9. *Outline of procedure.* We shall define a function $\psi(y)$ by means of different analytic functions over successive y intervals. The function $\psi(y)$ will later be modified to give our final function $\phi(y)$. Let the curve $z = \psi(y)$ be drawn in the zy plane. For the $\psi(y)$ to be described (§ 10), the segments of the curve through $y = 0$, $y = h$ and $y = -h'$ will be parabolic, while the intermediate segments will be linear. The resulting $\psi(y)$ will be continuous but in general there will be a discontinuity of $\psi'(y)$ at each junction. We first obtain a $\psi(y)$ of this description such that the integral

$$\zeta(c) = 2 \int_0^{\bar{z}} \frac{dy}{(c^2 - \psi(y))^{1/2}}, \quad c^2 = \psi(\bar{z}), \tag{12}$$

defines a function $\zeta(c)$, which satisfies the requirements made on $\eta(c)$ in § 8. We shall then modify $\psi(y)$ so as to obtain a function $\phi(y)$ of class C'''' such that the corresponding function $\eta(c)$ has the same properties as does $\zeta(c)$.

10. The *function* $\psi(y)$. The $\psi(y)$ used will be such that $\zeta(0) = \pi$. Hence in choosing b, the upper limit of (4), in order to satisfy (1) of Case I or III (§ 8), we require that

$$b/3 < \pi < b/2,$$

while to satisfy (1) of Case II or IV, we require that

$$b/2 < \pi < b.$$

A value of b, once appropriately chosen for the case under consideration, is to remain fixed during the subsequent discussion.

In any case, in defining $\psi(y)$ for positive y, we begin by assuming that

$$\text{(A)} \quad \psi(y) = y^2, \quad 0 \leqq y \leqq a_1,$$

where a_1 is an arbitrary positive constant. We have

$$\zeta(c) = 2\int_0^c \frac{dy}{(c^2-y^2)^{\frac{1}{2}}} = \pi, \quad 0 < c^2 \leqq \psi(a_1).$$

Now,

$$\lim_{c\to 0} \zeta(c) = \pi, \text{ and accordingly,}$$

we set $$\zeta(0) = \pi.$$

Let us suppose for the present that we are dealing with Case I. To obtain one of the points satisfying (2) Case I, we next define

$$\text{(B)} \quad \psi(y) = 2a_1 y - a_1^2, \quad a_1 < y \leqq a_2,$$

where a_2 will be chosen in a manner to be explained in the subsequent discussion. It will be noted that $\psi'(y)$ is continuous at $y = a_1$. Using (A) and (B), we have

$$\zeta(c) = 2\int_0^{a_1} \frac{dy}{(c^2-y^2)^{\frac{1}{2}}} + 2\int_{a_1}^{\bar{z}} \frac{dy}{(c^2+a_1^2-2a_1y)^{\frac{1}{2}}}, \quad \psi(a_1) < c^2 \leqq \psi(a_2),$$

where $\psi(\bar{z}) = c^2$. Integrating and differentiating,

$$\text{(13)} \quad \begin{aligned} &\zeta(c) = 2\sin^{-1}(a_1/c) + (2/a_1)(c^2-a_1^2)^{\frac{1}{2}}, \\ &\zeta'(c) = 2/(c^2-a_1^2)^{\frac{1}{2}}\,[c/a_1 - a_1/c], \quad \psi(a_1) < c^2 \leqq \psi(a_2). \end{aligned}$$

Let $F(c)$ denote the right hand member of (13) for any $c > a_1$. The function $F(c)$ becomes positively infinite with c. Furthermore, $F'(c) > 0$. Now let c_2 and the preceding constant a_2 be successively chosen in accordance with the conditions

$$\begin{aligned} &b/2 < F(c_2) = \zeta(c_2) < b, \\ &\psi(a_2) = c_2^2. \end{aligned}$$

The linear segment (B) of $\psi(y)$ is thus terminated so that $\zeta(c)$ does not exceed $\zeta(c_2)$ and hence never equals b. The next linear segment of $\psi(y)$ will be so chosen that $\zeta(c)$ immediately decreases from $\zeta(c_2)$.

We next define

$$\text{(C)} \quad \psi(y) = \beta_2 + 2\mu_2 y, \quad \mu_2 > a_1, \quad a_2 < y \leqq a_3,$$

where β_2, μ_2, and a_3 are constants and in terms of $\psi(y)$ as defined by (B),

$$\psi(a_2) = \beta_2 + 2\mu_2 a_2.$$

This insures the continuity of $\psi(y)$ at $y = a_2$. Now

$$\zeta(c) = 2\int_0^{a_1} \frac{dy}{(c^2 - y^2)^{1/2}} + 2\int_{a_1}^{a_2} \frac{dy}{(c^2 + a_1^2 - 2a_1 y)^{1/2}} + 2\int_{a_2}^{\bar{z}} \frac{dy}{(c^2 - \beta_2 - 2\mu_2 y)^{1/2}},$$

$$\psi(a_2) < c^2 \leqq \psi(a_3), \quad \psi(\bar{z}) = c^2.$$

Integrating and differentiating

$$\zeta(c)/2 = \sin^{-1}(a_1/c_1) + (1/a_1)(c^2 - \psi(a_1))^{1/2} + (c^2 - \psi(a_2))^{1/2}(1/\mu_2 - 1/a_2), \tag{14}$$

$$\frac{\zeta'(c)}{2} = \frac{(c/a_1 - a_1/c)}{(c^2 - \psi(a_1))^{1/2}} + \frac{c(1/\mu_2 - 1/a_1)}{(c^2 - \psi(a_2))^{1/2}}.$$

Set the right hand member of (14) equal to $G(c)$ for $c^2 > \psi(a_2)$. Computation reveals that $G''(c) > 0$. As c^2 increases from $\psi(a_2)$, $G'(c)$ increases from $-\infty$ through 0 approaching $1/\mu_2$ as c^2 becomes infinite. Hence $G(c)$ has one and only one minimum at $c = c_m$, say. The constant a_3 which terminates the interval of definition of (C) is to be chosen sufficiently large so that $\psi(a_3) > c_m^2$. Hence the domain of definition of $\zeta(c)$ in (14) will include $c = c_m$. Another requirement on a_3 will follow.

Now c_m and $\zeta(c_m)$ will of course depend upon the value chosen for μ_2. The required continuity of $\psi(y)$ determines the value of β_2 which is to correspond to a given μ_2. Now

$$\frac{d\zeta(c_m)}{d\mu_2} = \frac{\partial\zeta(c_m)}{\partial c_m} \cdot \frac{dc_m}{d\mu_2} + \frac{\partial\zeta(c_m)}{\partial\mu_2} = \frac{\partial\zeta(c_m)}{\partial\mu_2},$$

since

$$\frac{\partial\zeta(c_m)}{\partial c_m} = \zeta'(c_m) = 0.$$

But by (14)

$$\frac{\partial\zeta(c_m)}{\partial\mu_2} = \frac{-2}{\mu_2^2}(c_m^2 - \psi(a_2))^{1/2} < 0.$$

Hence $\zeta(c_m)$ is a decreasing function of μ_2. It can be shown that as μ_2 is increased from a_1 to ∞, $\zeta(c_m)$ decreases from $\zeta(c_2)$ to 0. We shall now choose μ_2 so that

$$b/3 < \zeta(c_m) < b/2.$$

For $c > c_m$, $\zeta(c)$ increases steadily. Let c_3 and the previous constant a_3 be determined by the relations

$$\psi(a_3) = c_3^2,$$

where

$$G(c_3) = \zeta(c_3) = \zeta(c_2).$$

Then by using the definition (C) of $\psi(y)$ over the range $a_2 < y \leqq a_3$, $\zeta(c)$ rises to $\zeta(c_2)$. As before, the next linear segment will cause $\zeta(c)$ to decrease immediately. For $\psi(a_2) < c^2 < \psi(a_3)$, we have $\zeta(c) = b/2$, twice, once with $\zeta'(c) > 0$ and once with $\zeta'(c) < 0$.

The character of $\zeta(c)$ for the range $\psi(a_2) < c^2 \leqq \psi(a_3)$ is typical for the remaining linear segments of $\psi(y)$ to be chosen. To proceed we define

$$\text{(D)} \quad \begin{aligned} &\psi(y) = \beta_3 + 2\mu_3 y, \quad \mu_3 > \mu_2, \quad a_3 < y \leqq a_4, \\ &\psi(a_3) = \beta_3 + 2\mu_3 a_3 \end{aligned}$$

where β_3, μ_3 and a_4 are constants. As before, $\zeta(c)$ has one and only one minimum at $c^2 = c_m{}^2 > \psi(a_3)$. By exactly similar steps, μ_3 may be so chosen that

$$b/3 < \zeta(c_m) < b/2.$$

To terminate (D), choose c_4 and the previous constant a_4 in accordance with the relations

$$\begin{aligned} &\zeta(c_4) = \zeta(c_3), \quad c_4 > c_3, \\ &\psi(a_4) = c_4{}^2. \end{aligned}$$

For $\psi(a_3) < c^2 < \psi(a_4)$, we have $\zeta(c) = b/2$ twice, once with $\zeta'(c) > 0$, once with $\zeta'(c) < 0$.

We continue in this manner, using q_1, segments such as (C) until $\zeta(c) = b/2$ a total of $(q_1 + 1)$ times with $\zeta'(c) > 0$ and q_1 times with $\zeta'(c) < 0$, thus satisfying (2) and (3) of Case I. (§ 8).

For Case III, we can use so far exactly the same $\psi(y)$. For Cases II or IV, if we use $q_1 + 1$ segments like (C), our choice of b relative to $\zeta(0)$ makes $\zeta(c) = b/2$, a total of $q_1 + 1$ times with $\zeta'(c) > 0$ and $q_1 + 1$ times with $\zeta'(c) < 0$. In all cases, for the range of c for which $\zeta(c)$ is now defined, $\zeta(c)$ oscillates between $\zeta(c_2)$ and $\zeta(c_m)$ and hence never equal b or $b/3$.

11. *The function $\psi(y)$ (continued).* Returning to Case I, we now wish to obtain the required intersections with $\zeta = b$. After the last intersection of $\zeta(c)$ with $\zeta = b/2$, instead of allowing $\zeta(c)$ to stop at $\zeta = \zeta(c_2) = \zeta(c_3)$, etc., let us continue to use the last linear segment of $\psi(y)$ until $\zeta(c) > b$. Then by using a proper succession of p_1 linear segments defining $\psi(y)$, $\zeta(c)$ will intersect $\zeta = b$ a total of $p_1 + 1$ times with $\zeta'(c) > 0$ and p_1 times with $\zeta'(c) < 0$, in such a manner that $\zeta(c)$ oscillates between K_1 and K_2 where

$$\begin{aligned} &b/2 < K_1 < b, \\ &K_2 > b. \end{aligned}$$

For a $\psi(y)$ so defined, let $\psi(-y) = \psi(y)$.

If we are dealing with Case II, this part of the procedure is exactly the same. For Cases III or IV, it is merely necessary to add to a $\psi(y)$ thus defined, an extra linear segment like (C) for $y > 0$.

In Cases I and II, we now finish off $\psi(y)$ for $y > 0$ with a parabolic segment having a maximum at $y = h$, where h is an arbitrary constant greater than any value of y prior to this segment. We then choose $h' = h$ and $\psi(-y) = \psi(y)$.

More explicitly, let k be the largest value of y for which $\psi(y)$ has been previously defined. We set

$$\psi(y) = \alpha - \beta(h-y)^2, \quad k < y \leqq h + e \tag{15}$$

where e is a positive constant and α and β are so chosen that the curve $z = \psi(y)$ is continuous at $y = k$.

The contribution to $\zeta(c)$ arising from (15) for $c^2 < \psi(h)$, is

$$I(c) = \int_k^{\bar{z}} \frac{dy}{[c^2 - \alpha + \beta(h-y)^2]^{\frac{1}{2}}}, \quad \psi(k) < c^2 < \psi(h), \quad \psi(\bar{z}) = c^2.$$

Integrating,

$$I(c) = (1/\beta^{\frac{1}{2}}) \log \{(h-k)/m + [(h-k)/m)^2 - 1]^{\frac{1}{2}}\},$$
$$m^2 = (\alpha - c^2)/\beta.$$

It is readily proved that $I'(c) > 0$, so that for $\psi(k) < c^2 < \psi(h)$, $\zeta(c)$ continues to increase, becoming infinite as c^2 approaches $\psi(h) = \alpha$.

In cases III and IV, we take the curve $z = \psi(y)$ as defined prior to this paragraph and add a parabolic segment such as the preceding to the part of the curve $y > 0$ and a similar segment to the part $y < 0$.

Except for continuity in its first four derivatives, $\psi(y)$ satisfies the requirements of § 2. The constant γ there introduced is to be chosen in the following manner. For Cases I and II, let $\gamma = h = h'$. For Cases III and IV, we let γ be the largest value of y for which $\psi(-y) = \psi(y)$ in the function $\psi(y)$ which we have described.

THEOREM I. *A continuous function $\psi(y)$ defined by different analytic functions over successive y intervals, and satisfying (B) (C) (D) § 2, may be so chosen that*

$$\zeta(c) = 2 \int_0^{\bar{z}} \frac{dy}{(c^2 - \psi(y))^{\frac{1}{2}}}, \quad \psi(\bar{z}) = c^2,$$

has the properties assigned to $\eta(c)$ in § 8.

12. *A function $\phi(y)$ approximating $\psi(y)$.* Having specified a function

$\psi(y)$ such that $\zeta(c)$ has the properties of the function $\eta(c)$ described in § 8, it remains to define a function $\phi(y)$ of class C'''' such that the corresponding

$$\eta(c) = 2\int_0^{\bar{y}} \frac{dy}{(c^2 - \phi(y))^{\frac{1}{2}}}, \quad \phi(\bar{y}) = c^2,$$

has the same properties. This we proceed to do by setting $\phi(y) = \psi(y)$ outside of an arbitrarily small interval including each junction, $y = a_i$. In the neighborhood of $y = a_i$, $\phi(y)$ will be so chosen that $\phi'(y)$ increases steadily from the slope of the preceding linear segment to the slope of the succeeding linear segment. Then in this neighborhood $\phi(y) > \psi(y)$.

In order to compare $\eta(c)$ and $\zeta(c)$ we require the following lemma.

LEMMA 5. *Given the integral*

$$I = \int_{\bar{y}-e}^{\bar{y}} \frac{dy}{(\phi(\bar{y}) - \phi(y))^{\frac{1}{2}}},$$

in which

$$\phi'(y) > m, \quad \bar{y} - e < y \leqq \bar{y}.$$

Then,

$$I < 2(e/m)^{\frac{1}{2}}.$$

Proof. Using the law of the mean,

$$\phi(\bar{y}) - \phi(y) = \phi'(y^*)(\bar{y} - y) > m(\bar{y} - y)$$

where

$$y < y^* < \bar{y}.$$

Hence,

$$I < (1/m^{\frac{1}{2}}) \int_{\bar{y}-e}^{\bar{y}} \frac{dy}{(\bar{y} - y)^{\frac{1}{2}}} = 2(e/m)^{\frac{1}{2}}.$$

In comparing $\eta(c)$ and $\zeta(c)$, it will suffice to consider the difference produced by the modification in the neighborhood of a single junction, $y = a_i$. For brevity let

$$a_i - (e/2) = r, \quad a_i + (e/2) = s, \quad \psi'(r) = m.$$

We are choosing $\phi(y)$ so that

$$\begin{aligned} &\phi(y) = \psi(y), \quad y \leqq r, \quad s \leqq y; \\ &\phi(y) > \psi(y), \quad \phi'(y) > m, \quad r < y < s. \end{aligned} \tag{16}$$

Now for $y \leqq r$, since $\phi(y) = \psi(y)$, we have $\bar{y}(c) = \bar{z}(c)$ and $\eta(c) = \zeta(c)$. For $y \geqq s$,

$$\eta(c) - \zeta(c) = 2\int_r^s \frac{dy}{(c^2 - \phi(y))^{\frac{1}{2}}} - 2\int_r^s \frac{dy}{(c^2 - \psi(y))^{\frac{1}{2}}}.$$

Then

$$0 < \eta(c) - \zeta(c) < 2 \int_r^s \frac{dy}{(c^2 - \phi(y))^{1/2}}$$

$$\leqq 2 \int_r^s \frac{dy}{(\phi(s) - \phi(y))^{1/2}} < 4(e/m)^{1/2},$$

using (16) and Lemma 5.

For $r < y < s$,

$$\eta(c) - \zeta(c) = 2 \int_r^{\bar{y}} \frac{dy}{(c^2 - \phi(y))^{1/2}} - 2 \int_r^{\bar{z}} \frac{dy}{(c^2 - \psi(y))^{1/2}} .$$

Then

$$\eta(c) - \zeta(c) < 2 \int_r^{\bar{y}} \frac{dy}{(c^2 - \phi(y))^{1/2}} < 4[(\bar{y} - r)/m]^{1/2} < 4(e/m)^{1/2}.$$

Also

$$\zeta(c) - \eta(c) < 2 \int_r^{\bar{z}} \frac{dy}{(\psi(\bar{z}) - \psi(y))^{1/2}} .$$

While $\psi(y)$ has a discontinuity in its derivative at $y = a_i$, we find by a slight modification of Lemma 5 that

$$\zeta(c) - \eta(c) < 4(e/m)^{1/2}.$$

Hence, *by a proper choice of* e, $|\eta(c) - \zeta(c)|$ *may be made arbitrarily small, uniformly for all values of* c *in the interval*

$$-[\psi(-h')]^{1/2} + e \leqq c \leqq [\psi(h)]^{1/2} - e, \quad e > 0.$$

We now compare $\eta'(c)$ and $\zeta'(c)$ for a value of $c^2 > \psi(s) = \phi(s)$ for which $\zeta(c) = b$ or $b/2$. Let $c^2 - \phi(s) = k$. Now

$$\zeta'(c) - \eta'(c) = 2 \int_r^s \frac{c\,dy}{(c^2 - \phi(y))^{1/2}} - 2 \int_r^s \frac{c\,dy}{(c^2 - \psi(y))^{1/2}} .$$

Then by (16)

$$0 < \zeta'(c) - \eta'(c) < 2 \int_r^s \frac{c\,dy}{(c^2 - \phi(y))^{3/2}}$$

$$< 2c \int_r^s \frac{c\,dy}{(c^2 - \phi(y))^{3/2}} = \frac{2ce}{K^{3/2}} .$$

Hence *by a proper choice of* e, $\zeta'(c) - \eta'(c)$ *may be made arbitrarily small in the neighborhood of an intersection of* $\zeta(c)$ *with* $\zeta = b$ *or* $\zeta = b/2$.

As a result of the foregoing, e may be chosen sufficiently small so that the curve $z = \eta(c)$ intersects $z = b$ and $z = b/2$ the same number of times as does $z = \zeta(c)$, and $\eta'(c)$ and $\zeta'(c)$ have the same sign at corresponding intersections.

THEOREM 2. *A function $\phi(y)$, satisfying the requirements of § 2, may be so chosen that the function*

$$\eta(c) = 2\int_0^{\bar{y}} \frac{dy}{(c^2 - \phi(y))^{\frac{1}{2}}}, \quad \phi(\bar{y}) = c^2,$$

has the properties assigned to $\eta(c)$ in § 8.

13. *Conclusion.* As a result of Theorem 2 we have the important theorem

THEOREM 3. *A function $\phi(y)$ of class C'''' and a constant b may be so chosen that the integral*

$$\int_0^b [y'^2 - \phi(y)]\, dx$$

possesses extremals joining $(0, 0)$ and $(b, 0)$ of which M_i $(i = 0, 1, 2)$ are of type i, where M_0, M_1, and M_2 are arbitrary positive integers satisfying

$$(3). \qquad \begin{aligned} &M_0 \geqq 2, \\ &M_0 - M_1 + M_2 = 1. \end{aligned}$$

Finally,

THEOREM 4. *Between the numbers M_i corresponding to a set of extremals joining any two fixed points in an extremal-convex region S', there exist no relations independent of (3).*

By a similar procedure it is easily shown that the relations,

$$\begin{aligned} &M_0 \geqq 2, \\ &M_0 - M_1 = 1 \end{aligned}$$

are complete for the case $n = 1$.

HARVARD UNIVERSITY.

An Intrinsic Treatment of Poisson's Integral.*

By Fred W. Perkins.

Introduction. In many phases of mathematical work it is desirable to have the statements of fundamental propositions available in a form which is independent of any particular coördinate system. Such a property not only broadens the range of applicability of a formula, but often, by suggesting a simple intuitive interpretation, throws light on the meaning of a relation. A number of such interpretations of Poisson's integral have been given. The present paper generalizes these, and gives special attention to the case in which the circle is replaced by the half plane, where the integral can be given a particularly simple form.† It will appear that in general the representations given for Poisson's integral are valid without change of form for circle and half plane.

Section 1 is devoted to the study of the case of the half plane and to the consideration of a question of uniqueness. Section 2 gives several intrinsic formulations of Poisson's integral, embracing as special cases those given by Schwarz, Neumann, and Bôcher. Section 3 develops an intrinsic formula for the directional derivative of a harmonic function in terms of its boundary values. In Section 4 these results are used to generalize certain theorems regarding the oscillation of harmonic functions, while Section 5 is devoted to a discussion of the spatial analogues of some of these theorems of oscillation.

1. *A Formula Related to Poisson's Integral.* The solution of the first boundary problem of potential theory for a half plane is obtainable at once from results due to Evans and Bray ‡ and to Evans § by a conformal mapping

* Parts of this paper were presented to the American Mathematical Society, January 2, 1926, in two communications entitled "A Function Related to Poisson's Integral" and "On the Oscillation of Harmonic Functions (Second Paper)."

† The transformation of the circle into the half-plane has been used by Osgood (*Funktionentheorie*, 4th edition, Leipzig, 1923, p. 638) as a basis for the discussion of Poisson's integral. But full use has not been made of the geometry of the situation.

‡ "Sur l'intégrale de Poisson généralisée," *Comptes Rendus*, t. 176 (1923), p. 1042; "La Formule de Poisson et le Problème de Dirichlet," *ibid.*, p. 1368.

§ "Sur l'Intégrale de Poisson," *ibid.*, t. 177, p. 241.

of the half plane on the interior of the unit circle, and the use of Bôcher's interpretation of Poisson's integral.*

Let the half plane be taken as either the upper half or the lower half of the plane of x and y, and P be one of its points ($y \neq 0$). Let p be a variable point of the x-axis and ω' the angle from the perpendicular to the x-axis from P to the tangent at P of the circle through P and p orthogonal to the x-axis. Half this angle is the angle ω from the perpendicular to the x-axis at P to the line Pp. The results cited then enable us to state the following theorem:

THEOREM. *If $f(p)$ is any function of p, measurable (L) as a function of x, and bounded, the Lebesgue integrals*

$$u(P) = (1/2\pi) \int_{-\pi}^{\pi} f(p)\, d\omega' = (1/\pi) \int_{-\pi/2}^{\pi/2} f(p)\, d\omega,$$

where $d\omega$ and $d\omega'$ are counted positive in the direction of increasing x, define a function of P which is bounded and harmonic throughout the interior of either half plane, and assumes the boundary values $f(p)$ for approach along the circles ω' at almost all p. It is the only such function.

From Fatou's theorem † we have also the following result:

The function $u(P)$ approaches the boundary value $f(p)$ as P approaches p in any direction which makes a positive angle with the boundary, for almost all p; if $f(p)$ is continuous at p_0, $u(P)$ approaches the boundary value $f(p_0)$ at p_0 for the general two dimensional sense of approach.

Suppose that we restrict ourselves to the two dimensional sense of approach, using the expressions "assumes" or "approaches the boundary values" only in this sense. The theorem then yields the following statement:

If $f(p)$ is continuous almost everywhere, and bounded, the above formulas give us a function which is bounded and harmonic throughout the interiors of the half planes, and assumes the boundary values at all points of continuity. It is the only such function.

It should be noted that the uniqueness property does not hold in the form

---

* *Bulletin of the American Mathematical Society*, Ser. 2, Vol. 4 (1897-98), p. 424; *Annals of Mathematics*, Ser. 2, Vol. 7 (1906), p. 92. See also Osgood, *loc. cit.*, p. 635.

† Fatou, "Séries Trigonométriques et Séries de Taylor," *Acta Mathematica*, Vol. 30 (1906), pp. 335-400, see p. 345 and p. 357.

stated above * if the requirement that $f(p)$ be continuous almost everywhere on the boundary be dropped, as is shown by the following example. Moreover it is impossible, in this example, to remove the difficulty merely by redefining $f(p)$ on a point set of measure zero. It may also be noted that the function $f(p)$ here constructed is of class 1, in the sense of Baire. We shall find it convenient to study the problem with reference to the interior of a circle; the result can be extended to a half plane by inversion.

We construct a point set e on an arc of the circumference C which is perfect and nowhere dense, yet such that the measure of the points common to e and any interval with mid-point at a point of e shall be positive. This can be done, for instance, by starting with an open arc of C, removing the open quarter with center at the mid-point of the arc, removing in the same way the open sixteenths of the remaining arcs, removing in the same way the open sixty-fourths of the four then remaining arcs, and so on indefinitely. The set e shall consist of the points of the initial arc never so removed. There is no difficulty in verifying the fact that it has the stated properties.

We now define $f(p)$ as 1 on e and 0 on the rest of C. Let $v(P)$ be the function defined by Poisson's integral for the values $f(p)$ on C, the integral being taken in the sense of Lebesgue. This function is bounded and harmonic throughout the interior of C; it is positive throughout this region, but it assumes the boundary value 0 continuously at each point p of the boundary at which the function $f(p)$ is continuous, namely all the points of C except those in e. Thus $v(P)$ and 0 are two distinct functions harmonic within C and assuming the boundary value $f(p)$ at every point of continuity of that function.

Moreover, the function $f(p)$ cannot be so altered at the points of any set of measure 0 that it uniquely determines a bounded harmonic function approaching it at every point of continuity. For suppose that $f^*(p)$ is any second function differing from $f(p)$ only on a set of measure 0. Poisson's integral, formed for $f^*(p)$ approaches $f^*(p)$ at each point of continuity. But the same is true of the function 0, since the only points at which $f^*(p)$ is continuous are among those for which it vanishes.

---

* For a discussion of a modified form of the uniqueness theorem for Poisson's Integral, see Fatou, *loc. cit.*, see also Plancherel, *Bulletin des Sciences Mathématiques*, Ser. 2, Vol. 34 (1910), p. 111. Necessary and sufficient conditions on a harmonic function for representation by Poisson's integral have been obtained by Evans and Bray and by Evans, *loc. cit.* See also the book of the latter on *Logarithmic Potential—Discontinuous Dirichlet and Neumann Problems*, in the Colloquium series of the American Mathematical Society.

To see this we need to verify that $f^*(p)$ is discontinuous at all the points of e. Let p_1 be any point of e. Every interval with p_1 as center contains infinitely many points at which $f(p)$ and $f^*(p)$ each have the value 1. For every such interval contains a point set of positive measure at which $f(p) = 1$, and $f^*(p) \neq f(p)$ only on a point set of measure zero. On the other hand, every such interval contains intervals exterior to the set e, and hence infinitely many points at which $f(p)$ and $f^*(p)$ have the common value zero. Hence, f^* is discontinuous at p_1. Thus $f^*(p)$ can be continuous only at points outside e, where it differs from 0 only at the points of a set of measure 0, and so cannot be continuous at any point where it differs from 0.

2. *Interpretations of Poisson's Integral.* In this section we shall derive a number of extensions and reformulations of known interpretations of Poisson's integral, obtaining certain results independent of coördinate systems. We shall consider functions which are bounded and harmonic in either the interior or exterior of a proper circle, or in a half-plane. From the point of view of the geometry of inversion a straight line is simply a circle through the point at infinity. Since many of the theorems under discussion can be formulated in such a way as to be applicable to a region bounded by this special type of circle as well as to the ordinary region, we shall use the terms "circle" and "point" as in the geometry of inversion, unless the contrary is explicitly indicated.

We shall confine our attention to the case that the boundary values are piece-wise continuous. In this case Poisson's integral,* or the formula of the first theorem in the preceding section, gives the unique bounded function which is harmonic throughout the plane except on the circle under consideration and assumes continuously the given boundary values at all points where the latter are continuous. For brevity, we shall refer to this harmonic function as the one "determined" by the given boundary values.

Our first theorem is an extended form of Schwartz's interpretation of Poisson's integral.†

THEOREM I. *Let p, p' be the intersections of a fixed circle C and a variable circle c through two distinct fixed points P, Q, not on C. Let $f(p)$ be a piece-wise continuous function on C, and let $g(p) = f(p')$. Let u, v be the*

* Here absolute value signs should be used for the coefficient of the boundary values $f(p)$ in the integrand, in order that the integral may give the solution of the exterior as well as the interior problem.

† Schwartz, *Gesammelte Werke*, Vol. 2, p. 360.

harmonic functions determined by the boundary values $f(p)$, $g(p)$ respectively. Then

$$\begin{cases} u(P) = v(Q), \\ u(Q) = v(P). \end{cases}$$

Schwarz's Theorem is equivalent to the special case in which C is a proper circle and Q the point at infinity; for when it is recalled that the value of v at infinity is the same as its value at the center of C, namely the average of the function $f(p')$, the theorem tells us that the value of u at P is the average of the values obtained by transferring the boundary value at p to the point p' on C collinear with P and p. Schwarz stated this result merely for the case in which P is inside C, but his demonstration can readily be extended to cover the case in which P lies outside C. His proof is based upon the formula for Poisson's integral. Another proof, free from computation, is given below.

In the case that C is a proper circle, and Q the point at infinity, we infer from a well-known theorem that the product $\overline{Pp} \cdot \overline{Pp'}$ has a constant value, k^2, as p traces C. If P lies outside C, this means that p, p' are conjugate points with respect to the circle Γ with center at P and radius k. If we invert in Γ the function u, finite and harmonic in C with boundary values $f(p)$ *on* C, is carried into a function v, bounded and harmonic within C, with boundary values $g(p) = f(p')$. Furthermore P is carried into the point at infinity, whence $u(P) = v(\infty)$, i. e. the average of the function $g(p)$. If P is an interior point of C, a rotation about P through an angle π followed by an inversion in Γ carries p into p', P into the point at infinity, and u into a function v bounded and harmonic exterior to C with the boundary values $g(p)$, whence the desired result follows.

The generalization to the theorem as stated above is obtained immediately by making an inversion with center at Q. Since Q does not lie on C, C will always be carried into a proper circle by this transformation; u, v are carried into two functions, u', v' such that

$$\begin{cases} u'(P') = v'(\infty), \\ u'(\infty) = v'(P'), \end{cases}$$

where P' is the point corresponding to P. Inverting again so as to return to the original figure, and using the fact that piece-wise continuous boundary values determine a bounded harmonic function uniquely, we obtain the desired result.

We note incidentally, as a limiting form of the above theorem, that if the boundary values on C are interchanged at each pair of points in which it is cut

by a variable circle tangent to a fixed line at a fixed point P, not on C, then the corresponding harmonic functions have the same value at P. The truth of this is immediately apparent on making an inversion with center at P.

THEOREM II. *Let P, Q be any two distinct fixed points not on the fixed circle C. Let ψ be the directed angle * from some fixed direction at P to that directed arc $\overset{\frown}{Pp}$ of the circle c, determined by P, Q and the variable point p on C, which does not pass through Q. Let u be the harmonic function determined by the piece-wise continuous boundary values $f(p)$ on C. Then*

$$u(P) = (1/\pi) \int_C f(P)\, d\psi \pm u(Q),$$

the upper or lower sign being chosen according as P, Q lie in the same region or different regions bounded by C. The integration in each case is taken in such a sense that the region containing P lies on the left.

We discuss first the case in which Q is the point at infinity, and C, therefore, a proper circle.

Here the theorem is obviously true if P is chosen as the center of the circle, so we may exclude this case from further consideration. We take as the fixed direction from which to measure ψ the direction from P to O, the center of C. Let p_0 be the first point in which the ray $\overrightarrow{PO}$, when traced in the positive sense from P, meets C, and let p_0' be the other point of intersection. Let p' be the second intersection of C and the line Pp. Denote the directed angle from Op_0 to Op by ϕ, that from Op_0' to Op' by ϕ'. In all cases we have, for suitable choices of the determinations of ϕ, ϕ', the relation

$$2\psi = \phi + \phi'.$$

Hence we have, according as P lies outside or inside C, either

$$(1/\pi) \int_C f(p)\, d\psi = (1/2\pi) \int_0^{-2\pi} f(p)\, d\phi + (1/2\pi) \int_0^{2\pi} f(p)\, d\phi',$$

or

$$(1/\pi) \int_C f(p)\, d\psi = (1/2\pi) \int_0^{2\pi} f(p)\, d\phi + (1/2\pi) \int_0^{2\pi} f(p)\, d\phi'.$$

---

* We may choose the determination of ψ arbitrarily, except for the condition that $\psi(p) \to \psi(q)$ as $p \to q$ for all points q on C, excepting one point (in the case that ψ is increased by 2π when p traces C in a suitable sense). A similar remark applies to θ in Theorem II-A below.

If P and Q are finite points and their roles are interchange the new angle ψ' replacing ψ may be so chosen that $\psi' = -\psi$. This fact suggests as a definition of ψ, in the case in which P is at infinity, the angle obtained by interchanging P and Q reversed in sign. With this convention the theorem is true in this special case.

Now

$$(1/2\pi)\int_0^{-2\pi} f(p)\,d\phi = -(1/2\pi)\int_0^{2\pi} f(p)\,d\phi = -u(\infty),$$

and

$$(1/2\pi)\int_0^{2\pi} f(p)\,d\phi' = u(P).$$

Hence,

$$u(P) = (1/\pi)\int_C f(p)\,d\psi \pm u(\infty)$$

according as P lies outside or inside * C.

Now the expression $(1/\pi)\int_C f(p)\,d\psi$ (as defined in the general case) has a meaning which is independent of an inversion. That is, if we carry the function $f(p)$ on C into a function $f'(p')$ on C' by an inversion, carrying also P, Q into P', Q' respectively, then it follows that this integral formed for one figure has exactly the same value as it has when formed with reference to the other. (The reversal of the sign of angles is compensated by the convention in regard to the sense of integration.) Hence, the general theorem follows by inversion from the special case treated above.

It will be noted that this result includes Bôcher's interpretation of Poisson's integral as the special case that P, Q are conjugate points with respect to C.

The following is another formulation of the same result, which is more convenient for some purposes:

THEOREM II-A.† *Let P, Q be any two distinct points not on the circle C,*

* This result, in the case that P lies inside C is essentially Neumann's interpretation of Poisson's integral, *Abelsche Integrale*, 2d edition, (1884) p. 410 ff. Neumann's proof is based on the method of the arithmetic mean. It will be noted that when P lies inside C the integral on the right represents simply the average of the boundary values on C as they appear to an observer at P. In the most general case in which P, Q lie in different regions bounded by C a similar meaning may be attached to the integral if we think of light as being refracted so that it comes to P along arcs of circles passing through Q. This device is similar to one used by Bôcher in the *Annals* paper cited above.

† This theorem has been given, for the special case in which P and Q are interior points of a proper circle, by Darboux, "Un peu de Géométrie à propos de l'Intégrale de Poisson," *Bulletin des Sciences Mathématiques*, 2d series, Vol. 34, (1910), p. 287 ff. (see particularly formula 12, on page 294). It should be noted that the angle *AMB* appearing in the form of the theorem there given must be interpreted as a *directed* angle. Failure to take account of this leads to inaccuracies in the application of the theorem on pages 294 and 295, although the final result, which we shall discuss under Theorem I of Section 4, is correct (the angles *AEB* and *AHB* being essentially positive).

and denote by θ the angle from $\overline{pQ}$ to $\overline{pP}$. Let u be the harmonic function determined by the piece-wise continuous boundary values $f(p)$ on C. Then

$$u(P) = (1/\pi) \int_C f(u)d\theta \pm u(Q)$$

according as P, Q lie in the same region or different regions bounded by C. The integration is to be taken in such a sense that the region containing P lies on the left.

The proof of the theorem for the case in which P and Q are finite points consists merely in noting that if the angle ψ of the preceding theorem is measured from the directed line $\overline{QP}$, then ψ is equal to θ, inasmuch as each is equal to one-half of that directed arc $\breve{QP}$ of c which does not go through p, measured in terms of the radius.* In each case, the angle is positive or negative according as the sense from Q to P on this arc is counter-clockwise or clockwise.

If one of the points P or Q is chosen as the point at infinity the corresponding ray $\overline{pP}$ or $\overline{pQ}$ is indeterminate; it is readily verified, however, that the theorem still holds if the ray in question is understood to have a fixed direction as p traces C.

3. *A Representation of the Derivative of Certain Harmonic Functions.*

THEOREM. *Let p be a variable point on a fixed circle † C, and c the circle through p tangent at the fixed point P, not on C, to the fixed directed line L. Let the curvative of c be represented by $\pm K(p)$, the sign being chosen so as to agree with the sign of that determination of the angle from L to the ray $\overline{Pp}$ which is numerically not greater than π. Let u be the harmonic function determined by the piece-wise continuous boundary values $f(p)$ on C. Then the derivative of u in the direction of L is given by*

$$[du/d\xi]_P = (1/2\pi) \int_C f(p)dK(p),$$

the sense of integration being so chosen that the region containing P lies on the left.

---

* The relation is true when c goes through the point at infinity inasmuch as θ and ψ are continuous on C.

† In particular, C may pass through the point at infinity. The conventions adopted at the beginning of Section 2 regarding the use of this and certain other terms are retained in this section.

We note that this representation of the derivative is independent of coördinate systems.

Let Σ be an arc on C lying entirely in the finite part of the plane.* Divide Σ into n arcs Δs_i, $P_{i-1}\breve{P}_i$, $i = 1, 2, \cdots, n$, the points of division being numbered in their order on Σ, traced in the positive sense.† We choose the points of division so as to include such points of discontinuity of $f(p)$ as lie on Σ.

We shall denote by f_i the value of $f(p)$ at the mid-point of Δs_i and by U_i the harmonic function determined by the boundary values f_i on Δs_i and 0 elsewhere on C.

Let P' be a point at a small distance $\Delta\xi$ from P on the line L, the direction $\overline{PP'}$ being the positive direction, ξ, on L. Denote by θ the angle ‡ from $\overline{pP}$ to $\overline{pP'}$.

By Theorem II-A of Section II, we know that if $\Delta\xi$ is sufficiently small to insure that P, P' lie in the same region bounded by C, then

$$U_i(P') - U_i(P) = (1/\pi)\int_C f_i\, d\theta.$$

Hence, denoting the value of θ at $P = P_i$ by θ_i we have

$$\frac{U_i(P') - U_i(P)}{\Delta\xi} = \frac{f_i(\theta_i - \theta_{i-1})}{\pi\Delta\xi}.$$

Excluding for the moment the special case that one of the ends of Δs_i lies on L, denote the radius of the circle through P, P' and p_i by r_i. Then

$$\Delta\xi = 2r_i \sin |\, \theta_i \,|,$$

and

$$\frac{U_i(P') - U_i(P)}{\Delta\xi} = \frac{f_i}{2\pi}\left[\frac{1}{r_i}\,\frac{\theta_i}{\sin |\, \theta_i \,|} - \frac{1}{r_{i-1}}\,\frac{\theta_{i-1}}{r_{i-1}\sin |\, \theta_{i-1} \,|}\right].$$

We note that this expression for the difference quotient is valid even in the previously excluded case that p_i or p_{i-1} lies on L, provided we agree to replace the corresponding fraction $1/r_i$ or $1/r_{i-1}$ by 0.

When $P' \to P$, the fraction $1/r_i$ approaches as a limit the curvature of the

---

* If C is in particular a proper circle, Σ may be the whole of C.

† That is so that the region (bounded by C) containing P lies on the left.

‡ The determination should be chosen at some arbitrarily chosen point p, on Σ but not on L, so that if p is held fast and P' allowed to approach P, then $\lim \theta = 0$. This determination is then extended continuously along C.

circle through p_i tangent to L at P. In all cases we obtain, on noting that θ and $K(p)$ have the same sign,

$$[dU_i/d\xi]_P = (f_i/2\pi)\ [K(p_i) - K(p_{i-1})] = (f_i/2\pi)\Delta K_i.$$

We introduce now the harmonic function $U^*(P)$ determined by the boundary values $f(p)$ on Σ, 0 on the remainder of C.

Allow n, the number of the arcs Δs_i on Σ to become infinite in such a manner that the length of the largest arc Δs_i approaches zero as a limit. On C, and consequently throughout each of the regions bounded by C, the difference $U^*(P) - \sum_{i=1}^{n} U_i$ ultimately becomes uniformly less in absolute value than any preassigned positive ϵ. From this, we infer that $(d/d\xi)\,[\sum_{i=1}^{n} U_i]_P$ converges to $[dU^*/d\xi]_P$. On the other hand, we know that

$$[(d/d\xi)\ (\sum_{i=1}^{n} U_i)]_P = (1/2\pi) \sum_{i=1}^{n} f_i\,\Delta K_i,$$

whence

$$[dU^*/d\xi]_P = (1/2\pi) \int_{\Sigma} f(P)\,d\,K(P),$$

the sense of integration being the positive sense on Σ.

This virtually completes the proof for the case in which C is a proper circle, inasmuch as Σ may here be chosen as the whole of C. If, on the other hand, C extends to infinity, we must examine the behavior of U^* when the ends of Σ are allowed to recede indefinitely in opposite directions on C. It is sufficient to show that in some neighborhood of the point P the function $U^*(Q)$ converges uniformly to $U(Q)$. Throughout any finite connected region enclosing P but containing no points of C, the angle subtended at Q by the ends of Σ can be made to differ from π by less than any preassigned positive quantity, by requiring merely that the ends of Σ lie beyond certain fixed points the positions of which depend on the region S, but not on the choice of Q within S. The desired result now follows immediately from the first theorem of the paper.

4. *On the Oscillation of Harmonic Functions in the Plane.* This section contains applications of the above results in obtaining extensions of theorems on the oscillation of harmonic functions.* The extensions consist largely in

* See Schottky, "Ueber die Wertschwankung der harmonischen Funktionem zweier reellen Veränderlichen und der Funktionen eines Complexen Arguments," *Journal für die Mathematik*, Vol. 117 (1896-1897), p. 225 ff.; Koebe, "Ueber das Schwarzsche Lemma und einige damit zusammenhängende Ungleichheitsbeziehungen der Potentialtheorie und Funktionentheorie," *Mathematische Zeitschrift*, Vol. 6 (1920), p. 52 ff.;

a reformulation of the theorems which renders possible their utilization under less simple conditions in the plane of the geometry of inversion.*

THEOREM I.† *Let P, Q be any two distinct fixed finite points in the same region S bounded by a circle C. Denote by p_1, p_2, the points of contact with C of the two circles through P, Q tangent to C, and by θ_1, θ_2, the directed angles $\sphericalangle Qp_1P$ and $\sphericalangle Qp_2P$, respectively. Let u be any function bounded and harmonic in S; then*

$$| u(P) - u(Q) | \leqq (| \theta_2 - \theta_1 | / \pi)\, D(u, C)$$

where $D(u, C)$ means the oscillation‡ of u on C.

If, further, u is continuous on C, but not a constant, then the upper sign always holds, but the inequality is the best which is applicable to all such functions.

This theorem is a generalization of a theorem given by Darboux, also discussed in the author's paper in the *Annals of Mathematics* cited above. In the case there treated—that in which P and Q are interior points of a proper circle C—one of the angles θ_1 and θ_2 is positive and the other negative, and so $| \theta_2 - \theta_1 |$ is the sum of the numerical values of the angles θ_1 and θ_2, which we represented by $A_1 + A_2$ in our earlier notation.

Jensen, "Investigation of a class of fundamental inequalities in the theory of analytic functions," *Annals of Mathematics*, Vol. 21 (1919), pp. 1-29; Darboux (*loc. cit.*), and Schwarz, *Gesammelte Abhandlungen*, Vol. II, p. 361.

Other properties of the oscillation of harmonic functions, of a type somewhat different than those studied here are given by Bouligand in a paper entitled "Sur la continuité et les approximations en dynamique des liquides," which is to appear in a forthcoming number of the *Journal de l'Ecole Polytéchnique*.

Further references to the literature will be found in the author's paper "On the Oscillation of Harmonic Function," *Annals of Mathematics*, 2d. Series, Vol. 27 (1925), pp. 159-170.

* We therefore use the terminology of the geometry of inversion, as in Sections 2 and 3.

† Since the publication of my paper in the *Annals of Mathematics*, mentioned above, I have learned that this result was given by Darboux (*loc. cit.*) for the case in which P and Q are interior points of a proper circle C, and the boundary values are piecewise continuous. In regard to the discussion given by Darboux see the footnote to Theorem II-A of Section 2 of the present paper.

‡ If u is discontinuous on C the oscillation of u on C is defined as the difference between the upper and lower bounds of u in S.

In the first part of the proof of this case previously treated, a relation is obtained which can be written in the form

$$| u(P) - u(Q) | \leqq | U(P) - U(Q) | D(u, C)$$

where U is the function bounded and harmonic within the proper circle C, assuming the boundary values 1 and 0 respectively on the two arcs of C determined by the points we now call p_1 and p_2. Furthermore, if u is continuous on C, but not a constant, the upper sign holds, but the inequality is the best which is applicable to all such functions. That these relations hold in the cases in which the region of definition of u is the exterior of a proper circle, or a half-plane, can be verified immediately by inversion. The general theorem under consideration can now be obtained by the use of Theorem II-A of Section 2 of this paper.

We shall now give a second proof of this theorem which is independent of the earlier demonstration.

We note that θ, which is continuous on C, cannot have a maximum or minimum except at p_1 and p_2, for at any other point p_0 the circle c through P, Q, and p_0 cuts C at an angle not zero, which clearly implies that θ either increases or decreases when p passes through the point p_0. Moreover, since C does not go through P and Q, θ cannot be constant throughout any arc of C. From these facts it follows that θ varies monotonically when p traces either of the arcs * $\overset{\smile}{p_1 p_2}$ of C. For, assuming that this is not true, we infer that there exist on γ, one of these arcs, two points r_1 and r_2 such that p_1, r_1, r_2, p_2 occur in the order named and the quantities † $\theta(p_1) - \theta(p_2)$ and $\theta(r_1) - \theta(r_2)$ do not have the same sign. From the continuity of θ it follows that there are two distinct points r_1' and r_2' on γ such that $\theta(r_1') = \theta(r_2')$, whence we deduce that θ is either constant on at least a part of the arc $\overset{\smile}{r_1' r_2'}$ or else attains a maxium or minimum value, in the strict sense, at some interior point of the arc $\overset{\smile}{r_1', r_2'}$. But we know that neither of these situations is possible, so we conclude that θ is monotonic on each arc $\overset{\smile}{p_1 p_2}$ of C.

---

* This is true even if C passes through the point at infinity. In such a case it is convenient to consider the behavior of the function θ on the corresponding arc of a proper circle obtained from C by inversion. In the new figure the function θ does not in general have the same geometrical meaning as before, but this is of no importance in the part of the proof now under consideration.

† We denote by $\theta(p)$ the value of θ at the point p of C.

Let u be any function harmonic in S and continuous on the boundary C.* Using the fact that θ is monotonic on each of the arcs $\overset{\smile}{p_1 p_2}$, we have

$$u(P) - u(Q) = (1/\pi)\int_C f(p)\,d\theta = (1/\pi)\{f(p')[\theta_2 - \theta_1] + f(p'')[\theta_1 - \theta_2]\},$$

or
$$u(P) - u(Q) = (1/\pi)[\theta_2 - \theta_1]\,[f(p') - f(p'')],$$

where p' is a point of one of the arcs $\overset{\smile}{p_1 p_2}$ and p'' a point of the other. From this it follows that

$$|\, u(P) - u(Q)\,| \leqq (|\,\theta_2 - \theta_1\,|/\pi) D(u, C).$$

Now $f(p')$ can be the maximum or minimum value of $f(p)$ on the corresponding arc $\overset{\smile}{p_1 p_2}$ only if $f(p)$ is constant on that arc; a similar remark applies to $f(p'')$. Since $f(p)$ is continuous on C it is not possible that $f(p')$ and $f(p'')$ should each be an extremum of $f(p)$ on C unless $f(p)$ is constant on C. Consequently in the relation given above the lower sign can hold (for functions u continuous on C) only in the trivial case in which u is a constant. That the inequality is the best applicable to all functions u harmonic in S and continuous on the boundary is readily verified by considering the harmonic function determined by the boundary values given below:

$$\begin{cases} f(p) = 0 \text{ on one arc } \overset{\smile}{p_1 p_2}, \\ f(p) = 1 \text{ on a part, } \overset{\smile}{r_1 r_2}, \text{ of the remaining arc } \overset{\smile}{p_1 p_2}, \\ f(p) \text{ is continuous and monotonic on the remainder of } C \text{ joining continuously to the boundary values specified for the other portions of } C. \end{cases}$$

If, now, in the inequality above, the factor $|\,\theta_2 - \theta_1\,|/\pi$ is replaced by a smaller quantity, then we can choose r_1 and r_2 so that $\overset{\smile}{r_1 r_2}$ comprises such a large part of $\overset{\smile}{p_1 p_2}$ that the new inequality fails to hold for the harmonic function determined by the continuous boundary values given above.

The case in which u is bounded and harmonic in S, but not necessarily continuous on C, can be treated by introducing a region S' which includes the points P and Q, and which, together with the proper circle C' forming its boundary, lies entirely in the open region S. Since u is continuous on C', we may write

$$|\, u(P) - u(Q)\,| \leqq (|\,\theta_2' - \theta_1'\,|/\pi)\, D(u, C'),$$

using primes throughout for quantities associated with S'.

---

* If the point at infinity lies on C, we shall agree that the restriction of continuity on C requires that the relation $\lim f(p) = f(q)$ holds even when q is the point at infinity, as $p \to q$.

Let R be any point in the region Σ' bounded by C' complementary to S'. Consider the behavior of the function $\theta(R) = \sphericalangle QRP$ in Σ'. We note that $\theta(R)$ is harmonic at each finite point of Σ'; moreover, it is continuous at infinity, (if the point at infinity is a point of Σ'), and it is continuous on C'. It follows that, even if Σ' extends to infinity,* the function $\theta(R)$ attains its extreme values only on C'. These extrema are θ_1' and θ_2'.

If now the center and the radius of C' are allowed to vary continuously in such a way that S' expands so that any given point of S ultimately becomes and remains a point of S', then the extreme values of $\theta(R)$ on S approach those on S, and consequently $|\theta_2' - \theta_1'| \to |\theta_2 - \theta_1|$. Moreover $D(u, C') \to D(u, C)$; hence we have as the limiting form of our inequality

$$|u(P) - u(Q)| \leqq (|\theta_2 - \theta_1|/\pi)\, D(u, C).$$

It should be noted that when discontinuities on C are allowed, it is not necessary that u be a constant in order that the lower sign hold. Such is obviously the case for the harmonic function determined by the boundary values unity on one arc $\overset{\smile}{p_1 p_2}$, zero on the other.

THEOREM II.† *Let C_1 and C_2 be any two fixed circles having no point in common; denote by S_1 the region bounded by C_1 containing C_2, by S_2 the region bounded by C_2 containing C_1. Let u_1, u_2 be any functions other than constants which are bounded and harmonic in S_1 and S_2 respectively. The ratios $q_1 = D(u_1, C_2) / D(u_1, C_1)$ and $q_2 = D(u_2, C_1) / D(u_2, C_2)$ depending*

---

* This case can be treated by an inversion.

† This theorem is a generalization of a theorem on the oscillation of harmonic functions on concentric circles due to C. Neumann. See the author's paper in the *Annals of Mathematics* cited above, and the papers by Schottky and Koebe, also cited above.

The constant q may be identified with the Hilbert configuration constant for the multiply connected region bounded by C_1 and C_2, in the case in which these are proper circles. See the discussion of the Hilbert constant (generalized for multiply connected regions with finite boundary) by E. R. Neumann, "Zur Integration der Potentialgleichung vermittelst C. Neumanns Methode des arithmetischen Mittels," *Mathematische Annalen*, Vol. 56 (1903) p. 49 ff. See particularly pp. 107-112 inc., where the theory is applied to the region bounded by two proper circles. The value obtained in the case in which the circles are concentric is identical with the constant q of the theorem now under consideration. Since each constant is independent of an inversion, they are identical in the general case. Neumann gives a formula for the Hilbert constant for the region bounded by two non-concentric circles, depending on the difference in the dipolar coördinates of the circles.

on the arbitrary functions u_1 and u_2 respectively, have a common least upper bound q, determined by the choice of the fixed circles C_1 and C_2. There exist functions u_1, u_2 such that $q_1 = q_2 = q$. If, however, we make the further hypothesis that u_1 and u_2 are continuous on C_1 and C_2 respectively, then, although q_1 and q_2 have the same least upper bound q as before, this bound is not attained.

If C_1 and C_2 are proper circles of radii a_1 and a_2 respectively, and with their centers at a distance k from each other, then

$$q = (2/\pi) \sin^{-1} (2a_1a_2/|\, a_1^2 + a_2^2 - k^2 \,|).$$

If C_1 or C_2 is a straight line and the other a proper circle of radius a with center at a distance h from this line, then

$$q = (2/\pi) \sin^{-1}(a/h).$$

The quantity q may be given the following geometric interpretation: let C be any circle cutting C_1 and C_2 orthogonally. Draw the two circular arcs c_1 and c_2 tangent to one of the given circles C_1 and C_2 and intersecting the other in the same points as does C. The number q is the ratio to π of the angle between c_1 and c_2.*

The numbers q_1 and q_2 associated with a pair of circles C_1 and C_2 and a pair of functions u_1 and u_2 are unchanged by an inversion, and so their least upper bounds for all admissible functions u_1 and u_2 have the same property. To prove, therefore, that q_1 and q_2 have the same least upper bounds it is sufficient to show that C_1 and C_2 can be interchanged by an inversion T. This is always the case, for by an inversion T_1 we can carry C_1 into a straight line C_1' and C_2 into a proper circle C_2'. Now C_1' and C_2' are conjugate with respect to the circle C' with center at that point of C_2' at the greatest distance from C_1' and radius equal to the geometric mean of this distance and the diameter of C_2'. It follows that C_1 and C_2 are conjugate with respect to the circle C corresponding to C' by the transformation T_1^{-1}.

In determining the value of q we shall consider first the case in which C_1 is a straight line. Let P and Q be any pair of points on C_2. Draw the circles through P and Q tangent to C_1; denote by p_1 and p_2 their points of contact with C_1. Make an inversion † in the circle orthogonal to C_2 with

---

* If each of C_1 and C_2 is a proper circle, C may conveniently be chosen as a straight line through their centers; if one of C_1, C_2 is a straight line, C may be chosen as the straight line perpendicular to this through the center of the other. C_1, C_2 cannot both be straight lines since by hypothesis they have no point in common.

† If p_1 is the point at infinity, this inversion is unnecessary.

center at p_1. Now C_1 and C_2 are unaltered by this transformation; the points P and Q are carried into P^* and Q^*, equidistant from C_1, and so p_2^*, the inverse of p_2, is the foot of the perpendicular to C_1, drawn from the center of C_2. The function u_1 is transformed into a function u_1^*, bounded and harmonic in the same region. By the first theorem of this section we have

$$| u_1^*(P^*) - u_1^*(Q^*) | \leqq (| \sphericalangle Q^* p_2^* P^* | / \pi)\, D(u_1^*, C_1).$$

But $| \sphericalangle Q^* P_2^* P^* |$ is at most as great as ω, the angle between the tangents to C_2 drawn from the point p_2^*. It follows that the least upper bound of $D(u_1^*, C_2)/D(u_1^*, C_1)$, and so also of $D(u_1, C_2)/D(u_1, C_1)$ is $q = \omega/\pi$; this bound is attained if, and only if, u is allowed to be discontinuous on C. We thus obtain the evaluation of q for the case that one of the circles C_1 or C_2 passes through the point at infinity; we also deduce, by inversion, the geometric interpretation of q given in the last paragraph of the statement of the theorem.†

It remains to compute the value of q for two proper circles. We shall first consider the case in which one of these circles, say C_2, lies inside the other. We invert in the circle of radius $(2)^{\frac{1}{2}} a_1$ with center at that intersection of C_1 and the line of centers of C_1 and C_2 which lies nearer C_2.‡ By this transformation C_1 is carried into a straight line C_1' through the center of C_1, while C_2 is carried into a proper circle with center at a distance

$$h = a_1^2/(a_1 - a_2 - k) + a_1^2/(a_1 + a_2 - k) - a_1$$

from C_1', and of radius

$$a = a_1^2/(a_1 - a_2 - k) - a_1^2/(a_1 + a_2 - k).$$

Since the value of q is independent of an inversion we can compute q for the circles C_1 and C_2 by substituting these values of h and a for the formula

$$q = (2/\pi) \sin^{-1} (a/h),$$

which gives the value of q for C_1' and C_2'. We thus find that for two proper circles, of which one lies inside the other,

$$q = (2/\pi) \sin^{-1} [2a_1 a_2/(a_1^2 + a_2^2 - k^2)].$$

---

† This interpretation is readily established for the case considered in Neumann's theorem (see the paper by Koebe cited above). It can then be extended to the general case by inversion, using the fact that the circles C_1 and C_2 can always be transformed into concentric circles by an inversion.

‡ If C_1 and C_2 are concentric we may use as center any point of C_1.

In case C_1 and C_2 are mutually exterior, we invert in a circle of radius $(2)^{\frac{1}{2}}a_1$ with center at the intersection of C_1 and the line segment joining the centers of C_1 and C_2.

In this case we have

$$\begin{cases} h = a_1^2/(k - a_1 - a_2) + a_1^2/(k - a_1 + a_2) + a_1, \\ a = a_1^2/(k - a_1 - a_2) - a_1^2/(k - a_1 + a_2); \end{cases}$$

whence,

$$q = (2/\pi) \sin^{-1} [2a_1a_2/(k^2 - a_1^2 - a_2^2)].$$

In the last formula we have $a_1^2 + a_2^2 - k^2 < 0$, whereas, in the corresponding equation for the case that one of the circles, C_1 and C_2, lies inside the other, we have $a_1^2 + a_2^2 - k^2 > 0$. Hence, for any two proper circles we may write

$$q = (2/\pi) \sin^{-1} (2a_1a_2 / | a_1^2 + a_2^2 - k^2 |).$$

We note that, if $k = 0$ and $a_2 < a_1$, we have

$$q = (2/\pi) \sin^{-1} [2a_1a_2/(a_1^2 + a_2^2)] = (4/\pi) \tan^{-1} (a_2/a_1)$$

in accordance with the extended form of Neumann's theorem due to Koebe.*

THEOREM III.† *Let P be any fixed finite interior point of a region S bounded by a circle C. Let L be a fixed directed line through P. Let $\pm K_1$, $\pm K_2$ denote the curvatures of the two circles tangent to L at P and to the circle C, the sign in each case agreeing with that of the determination numerically not greater than π of the angle from L to the ray joining P to the point of contact with C. Let u be any function bounded and harmonic in S, and denote by $[du/d\xi]_P$ the derivative of u in the direction of L at P. Then*

$$| du/d\xi |_P \leqq (1/2\pi) | K_2 - K_1 | D(u, C).$$

The lower sign may hold for certain functions discontinuous on C. If, however, u is continuous on C, but not a constant, then the upper sign always holds. Moreover this is the best inequality for the directional derivative applicable to all such functions.

The expression on the right is, in reality, independent of the direction ξ.

---

* See the paper in the *Mathematische Zeitschrift*, cited above. Another proof is given in the author's paper in the *Annals of Mathematics*, also cited above.

† This theorem is an extension of that given in § 3 of the author's paper in the *Annals*, cited above.

If C is a proper circle of radius a, and P is at a distance ρ from O, the center of C, then the above relation may be written in the form *

$$| du/d\xi |_P \leqq (2/\pi)\,(a/|\, a^2 - \rho^2 \,|)\, D(u, C).$$

If C is a straight line and P is at a distance δ from C, then the relation takes the form

$$| du/d\xi |_P \leqq D(u, C)/\pi\delta.$$

The function $K(p)$, introduced in the third section of this paper, is continuous and monotonic on each of the arcs of C bounded by p_1 and p_2, the points of contact with C of the two circles through P tangent to L and to C. It follows, then, from the theorem of that section that if u has the continuous boundary values $f(p)$ on C, we deduce from the relation

$$[du/d\xi]_P = (1/2\pi) \int_C f(p)\,dK(p)$$

the following equation:

$$[du/d\xi]_P = (1/2\pi)\,\{[K_2 - K_1]f(p') + [K_1 - K_2]f(p'')\},$$

or

$$[du/d\xi]_P = (1/2\pi)\,[K_2 - K_1][f(p') - f(p'')],$$

where p' is a point of one arc $\overset{\smile}{p_1p_2}$, and p'' a point of the other. We have then

$$| du/d\xi |_P \leqq (1/2\pi)\ |\ K_2 - K_1\ |\ D(u, C).$$

The other facts given in the first paragraph of the statement of the result are also obvious from the same type of reasoning as that used in the proof of the first theorem of this section.

We shall now discuss the geometrical problems of showing that the value of $|\, K_2 - K_1 \,|$ is independent of the direction ξ, and of expressing this quantity in terms of a and ρ, or of h, according to the nature of C. For any choice of P and ξ, the center of any circle tangent at P to a line L with the direction ξ

---

* An analogous relation for a regular function of a complex variable is virtually contained in a relation given by Schottky (*loc. cit.*). It is shown that if $\phi(x)$ is such a function in the circle with center at $x = o$ and radius R, and if $D(\phi)$ is the maximum value of $|\, \phi(y) - \phi(z)\,|$ for two arbitrary points y, z on the boundary, and r any positive number less than R, then

$$|\, \phi'(r)\,| \leqq D(\phi)\ .\ (2/\pi)\ .\ R/(R^2 - r^2).$$

This result can obviously be generalized. It will be noted that $D(\phi)$ is ordinarily greater than the oscillation on C of the real or pure imaginary part of $\phi(x)$.

lies on the line L' through P perpendicular to L. The center of any circle through P tangent to C lies on a certain conic. If C is a proper circle with center at O and radius a, then, according as P lies inside or outside C, this conic is an ellipse or a hyperbola with foci at O and P and major axis a; if C is a straight line, this conic is a parabola with P as focus and C as directrix. These facts follow immediately from elementary geometrical considerations, through the properties of the sum or difference of the focal radii in the case of the central conics.

In all cases except that in which P lies outside a proper circle C and L' has an intersection with the corresponding hyperbola lying on that branch farther from P, we know that K_1 and K_2, if different from zero, have opposite signs. For in these cases the centers of the circles through P tangent to L and to C, and therefore the points p_1 and p_2, lie on opposite sides of L. Here, then, $|K_2 - K_1|$ represents the sum of the reciprocals of the segments into which P divides the chord of the conic lying on L'. This is true when only one of the intersections is finite, provided the reciprocal of the half line now replacing a finite segment is interpreted as zero. On setting up a polar coördinate system with pole at P and prime vector along the line normal to the corresponding directrix but directed away from this line, we have as the equation of the conic,* or the part thereof involved,

$$r = em/(1 - e\cos\phi).$$

Hence, we see that the reciprocals under consideration may be written in the form

$$1/r_1 = (1 - e\cos\phi)/em,$$
$$1/r_2 = [1 - e\cos(\phi_1 + \pi)]/em,$$

or

$$1/r_2 = (1 + e\cos\phi)/em,$$

and so we have

$$|K_1 - K_2| = 2/em, \text{ or } |K_1 - K_2| = 4/l,$$

where l is the length of the latus rectum of the conic.

In the case that the conic is a hyperbola, the branch farther from P has as its equation with reference to the coördinate system used above,

$$r = -em/(1 + e\cos\phi).$$

---

* We use a familiar notation, regarding r as essentially positive. This equation cannot be applied in the special case that P coincides with O as the ellipse then becomes a circle. Here, however, the validity of the final result is immediately obvious.

In order that L' meet this branch,* it is necessary that L', (directed from P to this point of intersection) make an angle ϕ_1 with the prime vector such that $e \cos \phi_1 < -1$, which means that the same ray has a second finite intersection with the branch $r = em/(1 - e \cos \phi)$. Hence, K_1 and K_2 have the same sign in this case and

$$| K_1 - K_2 | = (1 - e \cos \phi_1)/em + (1 + e \cos \phi_1)/em,$$

or

$$| K_1 - K_2 | = 4/l.$$

Hence, we see that in all cases $| K_1 - K_2 |$ has the value $4/l$, independent of the direction ξ.

If C is a proper circle of radius a, and P is at a distance ρ from its center, the square of the semi-minor axis of the corresponding ellipse or hyperbola is $| a^2 - \rho^2 |/4$, and so

$$l = | a^2 - \rho^2 |/a, \text{ and } | K_2 - K_1 | = 4a/| a^2 - \rho^2 |.$$

Hence in this case the relation

$$| du/d\xi |_P \leqq (1/2\pi) | K_2 - K_1 | D(u, C)$$

can be written in the form

$$| du/d\xi |_P \leqq (2/\pi)(a/| a^2 - \rho^2 |) D(u, C).$$

In the case in which C is a straight line, and the point P is at a distance δ from C, the corresponding conic is a parabola and we have $l = 2\delta$. Hence in this case $| K_2 - K_1 | = 2/\delta$, and the relation

$$| du/d\xi |_P \leqq (1/2\pi) | K_2 - K_1 | D(u, C)$$

takes the form

$$| du/d\xi |_P \leqq D(u, C)/\pi\delta.$$

5. *On the Oscillation of Harmonic Functions in Space.* This section contains relations in space analogous to some of the results obtained for the plane in Section 4, though now given in a less general form. Given a function, bounded and harmonic in the interior† of a proper sphere S, we define $D(u, S)$

---

* By making suitable conventions as to signs, this case could be united with that previously discussed. It seems clearer, however, to give separate discussions.

† It will be noted that if $D(u, S)$ were similarly defined for a function bounded and harmonic *exterior* to S, then it would not necessarily agree with the usual definition of oscillation in the case that u is continuous on S. For example, if $u(P) = 1/r$ is the distance from the center of a unit sphere S to the point P exterior to S, we would have $D(u, S) = 1$ whereas the difference between the maximum and minimum

as the difference between the upper and lower bounds of u within S. The following result corresponds to Theorem I of the preceding section.

THEOREM. *Let S be a sphere of radius A, and P_1, P_2 any two distinct points within this sphere, at distances l_1, l_2, respectively, from the center O. Let σ be the sphere or plane which is the locus of all points Q such that*

$$\frac{\overline{P_1Q}}{\overline{P_2Q}} = \left(\frac{A^2 - l_1^2}{A^2 - l_2^2}\right)^{1/3}.$$

Denote by s the part of the surface S which lies within σ, (or, if σ is a plane, on either side of σ). Let P be any arbitrary point within S; denoting the distances from P to O and to an arbitrary point p of s by l and ρ respectively, set*

$$U(P) = (1/4\pi A) \iint_s \frac{A^2 - l^2}{\rho^3} \, dS.$$

If, now, $u(P)$ is any function bounded and harmonic within S, then

$$|u(P_1) - u(P_2)| \leqq |U(P_1) - U(P_2)| \, D(u, S).$$

If, further, $u(P)$ is continuous on S, but not a constant, then the upper sign holds, but this is the best appraisal of $|u(P_1) - u(P_2)|$ applicable to all such functions.

Let us first consider the case that u is continuous on S. If u is a constant, the validity of the first paragraph is obvious, so we may exclude this case from further consideration. We may, without loss of generality, suppose that u has the maximum and minimum values 1 and 0, respectively, on S. For if u' is any function harmonic within and continuous on S, having as extreme values on S the numbers M, m (where $m < M$), then

$$u = \frac{u' - m}{M - m}$$

is another function of the same sort with the maximum 1 and the minimum 0. If the theorem is true for the function u, it is also true for the function u', as can readily be seen from the relation

---

values of u on S is 0. In the treatment of these problems in space, it is not desirable to use the terminology of the geometry of inversion; by a sphere we mean always a proper sphere; by a point, a finite point, etc.

* It is readily shown by considerations of symmetry that in this case the value of $|U(P_1) - U(P_2)|$ is the same no matter which of the hemispheres of S determined by the plane σ is chosen as s.

$$u'(P_1) - u'(P_2) = [u(P_1) - u(P_2)]\, D(u', S).$$

Representing the boundary value of u at an arbitrary point p of S by $f(p)$, we have, from Poisson's integral

$$u(P_1) - u(P_2) = (1/4\pi A) \iint_S f(p)\, F(P_1, P_2)\, dS,$$

where $F(P_1, P_2) = \dfrac{A^2 - l_1^2}{\rho_1^3} - \dfrac{A^2 - l_2^2}{\rho_2^3} = \dfrac{A^2 - l_2^2}{\rho_1^3}\left[\dfrac{A^2 - l_1^2}{A^2 - l_2^2} - \dfrac{\rho_1^3}{\rho_2^3}\right]$,

ρ_1 and ρ_2 designating the values of ρ at P_1 and P_2 respectively.

A similar relation would hold in the case of a harmonic function constant within S, whence we infer that $F(P_1, P_2)$ is positive on a part of S and negative on a part of S. This means that σ always cuts S, and so s always exists. Furthermore P_1, P_2, lie on opposite sides of σ. If, in the inequality to be established, the points P_1, P_2 are interchanged, then neither side is affected in value, s being unaltered. Hence we may assume that P_1 lies on the same side of σ as does s, which implies that $F(P_1, P_2)$ is positive on s and negative on S-s (the remainder of s), points of the common boundary of these regions being disregarded. We have

$$u(P_1) - u(P_2)$$
$$= (1/4\pi a) \iint_s f(p)\, F(P_1, P_2)\, dS + (1/4\pi A) \iint_{S-s} f(p)\, F(p_1, p_2)\, dS.$$

Recalling that $0 \leqq f(p) \leqq 1$, we see that

$$\begin{cases} (1/4\pi A) \iint_s f(p)\, F(P_1, P_2)\, dS \leqq (1/4\pi A) \iint_s F(P_1, P_2)\, dS, \\ (1/4\pi A) \iint_{S-s} f(p)\, F(P_1, P_2)\, dS \leqq 0. \end{cases}$$

Moreover, the continuity of $f(p)$ on S implies that the lower sign cannot hold in both these relations. Consequently,

$$u(P_1) - u(P_2) < (1/4\pi A) \iint_s F(P_1, P_2)\, dS = U(P_1) - U(P_2).$$

A similar relation holds for the function $v(P) = 1 - u(P)$, which is finite and harmonic within S with the extreme values 0 and 1. This gives

$$-[u(P_1) - u(P_2)] < U(P_1) - U(P_2).$$

We therefore have

$$| u(P_1) - u(P_2) | < U(P_1) - U(P_2),$$

which implies the validity of the desired inequality for any function harmonic within and continuous on S.

To prove that this is the best possible inequality it is sufficient to show that, given any $\epsilon > 0$, there exists a $u(P)$ such that the relation

$$| U(P_1) - U(P_2) | - | u(P_1) - u(P_2) | < \epsilon$$

holds.

Denote by δ the distance from an arbitrary point p of s to the plane determined by the circle forming the boundary of s. Given any positive δ_0, consider the function $u_1(P)$, bounded and harmonic within S, determined by the continuous boundary values given below:

$$\begin{cases} f_1(p) = 0, \text{ on } S - s, \\ f_1(p) = \delta/\delta_0 \text{ when } p \text{ is a point of } s \text{ such that } \delta \leqq \delta_0, \\ f_1(p) = 1 \text{ on the remainder } s' \text{ of } s. \end{cases}$$

Assuming, as before, that s and P_1 lie on the same side of σ, we have

$$| U(P_1) - U(P_2) | - | u_1(P_1) - u_1(P_2) | \leqq (1/4\pi A) \int\int_{s-s'} F(P_1, P_2)\, dS$$
$$< \frac{1}{4\pi A} \cdot \frac{A^2 - l_1^2}{(A - l_1)^3} \cdot 2\pi A \delta_0 = \frac{(A + l_1)\delta_0}{2(A - l_1)^2}.$$

Hence, if we choose $\delta_0 = \dfrac{2(A - l_1)^2}{(A + l_1)}\, \epsilon$ we have the desired result.

We are now ready to discuss the case in which the restriction that u is continuous on S is replaced by the weaker condition that u is bounded within S. Make a transformation of space consisting of a shrinking toward the center of the sphere S, so that S is carried into sphere S' of radius $A' = \lambda A$, where $0 < \lambda < 1$. The function u, (which is not subjected to the transformation) is continuous on S', and so

$$| u(P_1') - u(P_2') | < | U'(P_1') - U'P_2') | D(u, S)$$

P_1' and P_2' being the points corresponding to P_1 and P_2, and U' the function associated with P_1', P_2', S' in the same way as is U with P_1, P_2, S. By considering the relations between A, ρ, l, ΔS and the corresponding quantities

with reference to the transformed figure, it is seen that at any pair of corresponding points P, P' we have

$$U'(P') = U(P).$$

Hence,

$$| u(P_1') - u(P_2') | < | U(P_1) - U(P_2) | D(u, S').$$

On letting $\lambda \to 1$, we have,

$$| u(P_1) - u(P_2) | \leqq | U(P_1) - U(P_2) | D(u, S),$$

the lower sign holding if, for instance, u is chosen as the function U.

THEOREM II. *Let S, S' be any two fixed concentric spheres of radii A, A' respectively (where $A' < A$). Let u be an arbitrary function bounded and harmonic within S, but not a constant. The ratios $D(u, S')/D(u, S)$ for all such functions u have a least upper bound q. The bound q is attained; if, however, the restriction is added that u is continuous on S, then the bound q is unaltered, but is not attained.*

The value of q in terms of the ratio $\mu = A'/A$ is

$$q = \frac{1}{\mu}\left[1 - \frac{1-\mu^2}{(1+\mu^2)^{\frac{1}{2}}}\right],$$

or in terms of the angle $\alpha = \tan^{-1}\mu$,

$$q = \sin(3\alpha/2)/\cos(\alpha/2)$$

or

$$q = [3 - 4\sin^2(\alpha/2)]\tan(\alpha/2).$$

The number q may be given the following geometric interpretation: draw the mutually perpendicular lines L, L' through O, the center of S and S', Let D be a point of intersection of L and S and D' a point of intersection of L' and S'. Denote by D'' the second point in which the line $\overline{DD'}$ cuts S'. With center at D and radius $\overline{DD''}$ draw a circular arc in the plane of L, L' cutting L in the point Δ. Then q is the ratio of the length $\overline{O\Delta}$ to A'.

Denote by P_1 and P_2 the points where an arbitrary one of the functions u assumes its extreme values of S'.

Let σ be the plane in which P_1, P_2 are symmetric, s the hemisphere of S bounded by σ and lying on the same side of σ as does P_1. Let

$$U(P) = \frac{1}{4\pi A} \iint_s \frac{A^2 - l^2}{\rho^3} \, dS,$$

(using the notation of the preceding theorem).

Represent by Q_1 and Q_2 the points in which the normal to σ at O meets S', Q_1, lying on the same side of σ as does P_1; then $U(P_1) \leqq U(Q_1)$, $U(P_2) \geqq U(Q_2)$ the lower signs holding in fact only when P_1 and Q_1 are identical. For, excluding this trivial case, let π be the plane of symmetry of P_1 and Q_1, σ' the reflection of σ in π, s_1 the larger part of s bounded by the intersections of σ and σ' with S, s_2 the remainder of s. Denote by U_1 the function given by Poisson's integral for the boundary value 1 on s_1, 0 elsewhere, and U_2 that for the boundary value 1 on s_2, 0 elsewhere. Then

$$\begin{cases} U_1(P_1) = U_1(Q_1), \text{ by symmetry,} \\ U_2(P_1) < U_2(Q_2), \text{ since each point of } s_2 \text{ is nearer } Q_1 \text{ than } P_1. \end{cases}$$

Hence $U(P_1) = U_1(P_1) + U_2(P_1) < U_1(Q_1) + U_2(Q_1) = U(Q_1)$.

Let $$V(P) = \frac{1}{4\pi A} \iint_{S-s} \frac{A^2 - l^2}{\rho^3} \, dS,$$

then $$\begin{cases} U(Q_2) = V(Q_1) = 1 - U(Q_1), \\ U(P_2) = V(P_1) = 1 - U(P_1), \end{cases}$$

and so $U(Q_2) - U(P_2) = U(P_1) - U(Q_1) < 0$, or $U(P_2) > U(Q_2)$.

The explicit formula for q in terms of A, A' can be obtained by integration from the relation

$$q = \frac{A^2 - A'^2}{4\pi A} \int_0^{\pi/2} \int_0^{2\pi} \left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3} \right) A^2 \sin\theta \, d\phi \, d\theta$$

where $\rho_1 = (A^2 - 2AA' \cos\theta + A'^2)^{\frac{1}{2}}$, $\rho_2 = (A^2 + 2AA' \cos\theta + A'^2)^{\frac{1}{2}}$.

This yields

$$q = \frac{A}{A'} - \frac{A^2 - A'^2}{A'(A^2 + A'^2)^{\frac{1}{2}}} \quad \text{or} \quad q = \frac{1}{\mu} - \frac{1 - \mu^2}{\mu(1 + \mu^2)^{\frac{1}{2}}} .$$

We note that we may write

$$q = \frac{A - \left(\dfrac{A^2 - A'^2}{(A^2 + A'^2)^{\frac{1}{2}}} \right)}{A'} .$$

Now $\overline{DD'} \cdot \overline{DD''} = A^2 - A'^2$; moreover, $\overline{DD'} = (A^2 + A'^2)^{\frac{1}{2}}$. Hence we see that the parenthesis in the numerator represents DD''. The interpretation for q given in the last paragraph of the theorem follows.

From this we may obtain readily the trigonometric formula for q. Drawing the line $D''\Delta$, it is readily shown that

$$\sphericalangle OD''\Delta = 3\alpha/2, \quad \text{and} \quad \sphericalangle O\Delta D'' = \pi/2 + \alpha/2;$$

hence on applying the Law of Sines to the triangle $OD''\Delta$ we have

$$q = \overline{O\Delta}/\overline{OD''} = \sin(3\alpha/2)/\cos(\alpha/2) = [3 - 4\sin^2(\alpha/2)]\tan(\alpha/2).$$

This completes the proof of the theorem.

HARVARD UNIVERSITY.

On the Invariant Combinants of Two Binary Quintics.*

By Thomas W. Moore.†

The question of the invariant combinants of two binary quintics considered in this paper presented itself in a study of the rational space quintic curve. Such a curve in a space of three dimensions is defined by means of four binary quintic forms giving the parameters of the sets of five points in which each of the four reference planes cuts the curve. Then if some projective specialization occurs in the curve it will be disclosed by some invariant or covariant condition imposed on these four quintics. In this way a study of the geometric properties of the curve can be converted into an investigation of the covariant theory of these four forms in question.

Two binary forms of the nth order are apolar if their bilinear invariant (nth ordered transvectant) vanishes. The apolarity relation is a linear condition on the coefficients, and it is known that there are $(n - r + 1)$ linearly independent forms of order n apolar to each of r such given forms. Hence there are two binary quintics apolar to the four given quintics defining the R_3^5. The curve may equally well be defined by these two quintics. The linear combination of these two forms in terms of a parameter determines what is called the fundamental involution on the curve.

This is interpreted geometrically by noticing that any plane (ξx) cuts the curve R_3^5 in five points whose parameters are given as roots of a form which is a linear combination of the four quintics defining the curve. Hence any plane section of the curve will be apolar to the two fundamental quintics, and a necessary and sufficient condition that five points of the curve be on a plane is that the binary quintic defining them be apolar to each of the two fundamental quintics. If $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are the five points and $f = a_\lambda^5$, $f' = a'_\lambda{}^5$ the two fundamental quintics, then the conditions are

$$a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} a_{\lambda_4} a_{\lambda_5} = 0, \qquad a'_{\lambda_1} a'_{\lambda_2} a'_{\lambda_3} a'_{\lambda_4} a'_{\lambda_5} = 0,$$

or if s_1, s_2, s_3, s_4, s_5 are the elementary symmetric functions of the five λ's:

* Presented to the American Mathematical Society, Feb. 26, 1927.
† National Research Fellow in Mathematics.

$$\left\{\begin{array}{l} a_0s_5 + a_1s_4 + a_2s_3 + a_3s_2 + a_4s_1 + a_5 = 0 \\ a'_0s_5 + a'_1s_4 + a'_2s_3 + a'_3s_2 + a'_4s_1 + a'_5 = 0 \end{array}\right\}$$

From this point of view the study of the rational quintic becomes a study of the system of invariant and covariant forms of the two binary quintics. Of especial interest are the covariant forms known as combinants. An invariant or covariant of any number of binary forms of the same order $f_1, f_2, \cdots, f_r$ is called a combinant if it be unaltered except as regards a factor independent of the forms when each form is replaced by a linear combination of the type $l_1f_1 + l_2f_2 + \cdots + l_rf_r$ in which the l's are constants.* The combinants are important geometrically because they are unaltered not only by a linear transformation applied to the variables, but also by any linear transformation effected on the coefficients. The natural geometric representation of the combinants is in the theory of rational curves, and the projective properties of the curve are defined by them.

The simultaneous system of covariant forms of two binary quintics has never been established. The complete systems of single binary forms up to the tenth order have been worked out by Gordan, Clebsch, von Gall, Sylvester and others. Gordan proved in general that the number of forms in either the complete system of a single form or of a simultaneous system of two forms is finite. The simultaneous systems of two forms as high as the fourth order were set up by Gordan first in his article "Die simultanen Systeme binärer Formen" in Vol. 2 of the *Mathematische Annalen*. For instance the system of two quartics is given on p. 275 of that volume.

We propose to concern ourselves here with the combinant forms of the two quintics, inasmuch as this phase arises from the geometrical question. One might be led to suspect that the systems of combinants of either the set of four quintics which are the coordinate forms of the quintic curve or the set of two apolar quintics are equivalent, since the curve may be given either way. This is indeed the case, as Meyer has shown.† In fact, by virtue of the relations existing between the fourth order determinants in one set and the second order determinants in the other we can secure one set of forms from the other, so far as coefficients are concerned.

The theory of the combinant forms has been the subject of extensive investigation by Gordan in a memorable article "Ueber Combinanten" published in Vol. 3 of the *Mathematische Annalen*. Here he proved the theorem

---

* Grace and Young, *Algebra of Invariants*, p. 314.

† W. F. Meyer, *Apolarität und Rationale Kurven*, § 11.

that the complete system of combinants of two binary forms of the same order is identical with the complete system of covariant forms of the joint set of the first, third, fifth, etc., transvectants of the two forms. These odd-ordered transvectants are combinant forms, and we remark besides that every invariant or covariant of a combinant is also a combinant of the two quantics.

E. Stroh called attention to the fact that such a system of covariant forms is not the most general of its type, due to the syzygies which exist between these odd-ordered transvectants. Let us suppose that the two given quantics are ϕ and ψ of the mth order. These odd-ordered transvectants are then $(\phi\psi)$, $(\phi\psi)^3$, $(\phi\psi)^5$, etc., of orders $2m-2$, $2m-6$, $2m-10$, etc., respectively. Now Stroh has shown * that such a complete system is the same as the complete system of a single binary form of order $2m-2$, the unique form apolar to the two given quantics of order m. This apolar form of order $2m-2$ is the form discussed by Gordan in his paper on resultants † and it vanishes identically if ϕ and ψ have two or more roots in common.

Consequently we know exactly the number of forms in such a system provided we know the number in the system of the form of order $2m-2$. For example consider the case of two quartics. The first and third transvectants are a sextic and a quadratic respectively. The apolar form is a sextic and hence there are just 26 covariant forms in the simultaneous system, and thus 26 combinants of the two quartics. This system was given by Stéphanos in the *Comptes Rendus.*‡

We are interested here in the complete system of combinants of two quintics, and to the author's knowledge little has been done on this so far. The first, third and fifth transvectants are an octavic, a quartic, and an invariant respectively. The single apolar form is an octavic, and so we have sixty-nine combinant forms (von Gall). Apparently the only work published on the combinants of two quintics is a short paper by Berzolari,§ in which he gives three syzygies connecting these odd-ordered transvectants above.

Let us denote the two quintics defining the fundamental involution by $Q = q_x^5$, $Q' = q'_x{}^5$. Then the first, third and fifth transvectants are respectively

---

* E. Stroh, *Mathematische Annalen*, Vol. 34, pp. 321 ff.

† P. Gordan, "Ueber die Bildung der Resultante zweier Gleichungen," *Mathematische Annalen*, Vol. 3 (1871), pp. 355-414.

‡ C. Stéphanos, "Sur le système complet des combinants de deux formes binaires biquadratiques," *Comptes Rendus*, Vol. 97 (1883), p. 27.

§ L. Berzolari, "Sui combinanti dei sistemi lineari di quintiche binarie," *Rendiconti di Palermo*, Vol. 7 (1893), p. 5.

$$f = a_x^8 = (qq')q_x^4 q'_x{}^4, \qquad \phi = \alpha_x^4 = (qq')^3 q_x^2 q'_x{}^2, \qquad \gamma = (qq')^5.$$

If we denote the apolar octavic by $F = F_x^8$, we have the following relations due to Stroh: *

$$(qq')q_x^4 q'_x{}^4 = (1/1200R^3)\ [8i_2 + (9/7)k^2 - (1/3)(Ai + 2BF)]$$
$$(qq')^3 q_x^2 q'_x{}^2 = -(1/200R^3)\ [(2/7)k_2 - i_k + (2/15)Ak]$$
$$(qq')^5 = (1/800R^3)\ [(1/3)A^2 - 2C]$$

where R is the resultant of the two quintics, and the forms on the right, all pertaining to F, will be explained later.

Berzolari's three syzygies are:

(I) $\quad (ff)^4 - (4/7)(f\phi)^2 + (1/15)\gamma f - (15/98)\phi^2 = 0$

(II) $\quad 2(ff)^6 + (15/7)(f\phi)^4 - (50/49)H_\phi - (1/3)\gamma \cdot \phi = 0$

(III) $\quad (ff)^8 + (5/7)i_\phi - (2/15)\gamma^2 = 0.$

The explicit forms of part of the forms above may be written down easily. We have

$$Q = q_x^5 = q_0 x_1^5 + 5q_1 x_1^4 x_2 + 10q_2 x_1^3 x_2^2 + 10q_3 x_1^2 x_2^3 + 5q_4 x_1 x_2^4 + q_5 x_2^5$$
$$Q' = q'_x{}^5 = q'_0 x_1^5 + 5q'_1 x_1^4 x_2 + 10q'_2 x_1^3 x_2^2 + 10q'_3 x_1^2 x_2^3 + 5q'_4 x_1 x_2^4 + q'_5 x_2^5.$$

Let us set $p_{ij} = q_i q_j' - q_j q_i'$ $(i, j = 0, 1, \cdots, 5)$ where the q_i are the actual coefficients. Then we have:

$$\begin{aligned} f = (qq')q_x^4 q'_x{}^4 = {} & p_{01}x_1^8 + 4p_{02}x_1^7 x_2 + 2(3p_{03} + 5p_{12})x_1^6 x_2^2 \\ & + 4(p_{04} + 5p_{13})x_1^5 x_2^3 + (p_{05} + 15p_{14} + 20p_{23})x_1^4 x_2^4 \\ & + 4(p_{15} + 5p_{24})x_1^3 x_2^5 + 2(3p_{25} + 5p_{34})x_1^2 x_2^6 \\ & + 4p_{35}x_1 x_2^7 + p_{45}x_2^8. \end{aligned}$$

$$\begin{aligned} \phi = (qq')^3 q_x^2 q'_x{}^2 = {} & (p_{03} - 3p_{12})x_1^4 + 2(p_{04} - 2p_{13})x_1^3 x_2 \\ & + (p_{05} + p_{14} - 8p_{23})x_1^2 x_2^2 + 2(p_{15} - 2p_{24})x_1 x_2^3 \\ & + (p_{25} - 3p_{34})x_2^4 \end{aligned}$$

$$\gamma = (qq')^5 = p_{05} - 5p_{14} + 10p_{23}.$$

The apolar octavic is best expressed as the bordered resultant, using Bézout's method of elimination.

---

* E. Stroh, *loc. cit.*, p. 326.

$$\begin{vmatrix} 5p_{01}, & 10p_{02}, & 10p_{03}, & 5p_{04}, & p_{05}, & x_2^4 \\ 10p_{02}, & 10p_{03}+50p_{12}, & 5p_{04}+50p_{13}, & p_{05}+25p_{14}, & 5p_{15}, & -4x_1x_2^3 \\ 10p_{03}, & 5p_{04}+50p_{13}, & p_{05}+25p_{14}+100p_{23}, & 5p_{15}+50p_{24}, & 10p_{25}, & 6x_1^2x_2^2 \\ 5p_{04}, & p_{05}+25p_{14}, & 5p_{15}+50p_{24}, & 10p_{25}+50p_{34}, & 10p_{35}, & -4x_1^3x_2 \\ p_{05}, & 5p_{15}, & 10p_{25}, & 10p_{35}, & 5p_{45}, & x_1^4 \\ x_2^4, & -4x_1x_2^3, & 6x_1^2x_2^2, & -4x_1^3x_2, & x_1^4, & 0 \end{vmatrix}$$

So according to the theory of Stroh we shall get the complete system of combinants of the quintics Q and Q' by considering the simultaneous system of covariants of f, ϕ, and γ. Since γ is invariant it may be neglected and we need consider only the joint system of the Jacobian octavic f and the quartic ϕ. This system is in turn equivalent to the single complete system of the apolar octavic. Now starting with the two forms f and ϕ it is our task to set up the joint system of covariants and reduce them to the least possible number by use of the three syzygies above due to Berzolari. We shall make use of a formula known as Gordan's series to effect this. Namely, if f, ϕ, ψ are three binary form of orders m, n, p respectively, we shall have the following:

$$\sum_i \frac{\binom{n-\alpha_1-\alpha_3}{i}\binom{\alpha_2}{i}}{\binom{m+n-2\alpha_3-i+1}{i}} \{(f,\phi)^{\alpha_3+i}, \psi\}^{\alpha_1+\alpha_2-i}$$

$$= (-1)^{\alpha_i} \sum_i \frac{\binom{p-\alpha_1-\alpha_2}{i}\binom{\alpha_3}{i}}{\binom{m+p-2\alpha_2-i+1}{i}} \{(f,\psi)^{\alpha_2+i}, \phi\}^{\alpha_1+\alpha_3-i}$$

represented schematically by

$$(f, \phi, \psi)\ (m, n, p;\ \alpha_1, \alpha_2, \alpha_3)$$

where $\alpha_2 + \alpha_3 \leq m, \quad \alpha_3 + \alpha_1 \leq n, \quad \alpha_1 + \alpha_2 \leq p,$

and either $\alpha_1 = 0 \quad$ or $\quad \alpha_2 + \alpha_3 = m.$*

The simultaneous system of two forms should contain ordinarily the complete system of each form alone together with the minimum number of such forms as are secured by transvection of a form in one system with a form in the other. But here the complete system of combinants can contain

* Grace and Young, p. 56.

only sixty-nine independent forms, and there are that many in the system of the Jacobian octavic alone. This Jacobian is a general octavic form and none of its forms will be reducible in the domain of its own coefficients. However certain covariants of this octavic are connected with the forms of the quartic ϕ and invariant γ by the three syzygies of Berzolari. Since the forms of the quartic are much simpler than those of the octavic, it appears expedient to attempt to express as many of the forms of the Jacobian octavic in terms of the covariants of the quartic as is possible. Normally one would expect to consider the invariants of the octavic first, for they are simpler to handle and of more immediate geometric interest.

Fundamentally then our object is to find a set of covariants involving forms of the quartic ϕ and the invariant γ and simpler cross-transvectants of f and ϕ and their lower degree forms with which to replace as many as possible of the forms in the system of the octavic f. Certain of the cross-transvectants appear in this procedure, and since we know the maximum number of possible forms, and as it will appear later in the case of the invariants that we get at least this number, we need not concern ourselves for the present with the problem of direct cross-transvection from the system of the octavic to that of the quartic, which would be somewhat of an endless task. Briefly, if we can show that we get all necessary forms from the system of the octavic alone, we need not go further. This then we shall consider first.

The complete system of the quartic ϕ contains five forms:

$$\phi = \alpha_x^4;\ H_\phi = H_{\phi_x}{}^4 = (\phi\phi)^2;\ t_\phi = (\phi H_\phi);\ i_\phi = (\phi\phi)^4;\ j_\phi = (\phi H_\phi)^4.$$

The complete system of the Jacobian octavic contains sixty-nine forms. It is given by von Gall in three articles in Vol. 17 of the *Mathematische Annalen*. The following table is taken from one of these articles (p. 149).

$k = (ff)^6$,	$i = (ff)^4$,	$H = (ff)^2$,	$f_k = (fk)^4$,	$f_3 = (fk)^3$, . . .,
$p = (fi)^2$,	$\theta = (fi)$,	$T = (Hf)$,	$\Delta = (kk)^2$,	$S = (Hi)$,
$f_{kk} = (f_k k)^4$,	. . . ,	$f_\Delta = (f\Delta)^4$,	$f''' = (f\Delta)^3$,	. . . ,
$\mathrm{T} = (k\Delta)$,	. . . ,	$f_{\Delta\Delta} = (f_\Delta\Delta)^4$,	. . . ,	. . . ,

ORDER.

Degree	0	2	4	6	8	10	12	14	18	Totals
1					f					1
2	$A=(ff)^8$		k		i		H			4
3	$B=(fi)^8$		f_k	f_3	f_2	f_1	p	θ	T	8
4	$C=(ki)^4$		Δ, i_k	i_3	i_2	H_3, i_1	H_2	H_1	S	10
5	f_{kk}	f_{k3}	f_{k2}, f_Δ	f_{k1}, f'''	f''	θ_k, p_3, f'		p_1		11
6	i_{kk}	i_{k3}	i_{k2}, i_Δ	i_{k1}, i''', T	i''	i'				9
7	$f_{k\Delta}$	$f_k''', (f\mathrm{T})^6$	$f_k'', (f\mathrm{T})^5$	$f_k', (f\mathrm{T})^4, \theta_{kk}$						8
8	$i_{k\Delta}$	$i_k''', (i\mathrm{T})^6$	$i_k'', (i\mathrm{T})^5$	$i_k', (i\mathrm{T})^4$						7
9	$f_{\Delta\Delta}$	$(f_k\mathrm{T})^4, f_\Delta''', \theta_{kkk}$	f_Δ''							5
10	$i_{\Delta\Delta}$	$(i_k\mathrm{T})^4, i_\Delta'''$								3
11		$(f_\Delta\mathrm{T})^4, \theta_{kkk2}$								2
12	⎭	$(i_\Delta\mathrm{T})^4$								1
Totals	9	14	13	12	6	7	3	3	2	69

In undertaking the reduction we shall confine ourselves to the invariants, but it will be necessary to reduce some intermediate covariant forms as well. The reduction process for the invariants of higher degree becomes very involved and for these the method of attack and the results will be sketched very briefly. An attempt will be made to outline the methods as clearly as possible for the lower degree forms.

We begin with the invariant $A = (ff)^8$. From Syzygy (III) we notice at once that it can be written in terms of γ^2 and i_ϕ. Then we may say that A is reducible in our sense; that γ and i_ϕ are fundamental.

The next invariant is $B = (fi)^8$, where $i = (ff)^4 = i_x^8$. From Syzygy (I) i is expressed in terms of $(f\phi)^2$, γf, ϕ^2. Then in the transvectant form $(fi)^8$ we replace i by each one of these and attempt a further reduction by Gordan's series. We shall frequently use the following notation for this:

$$i \equiv \gamma f,\ \phi^2,\ (f\phi)^2 \quad \text{or also} \quad i \equiv (1),\ (2),\ (3);$$

that is, each of the constituent parts will be numbered serially and if there are several component parts of either member of the original transvectant, the transvectant of any two members will be indicated by compounding the numbers. The system of indexing will become fairly self-explanatory as we proceed.

Then we turn to the reduction of the invariant B.

B1: $\gamma(ff)^8 = \gamma A \equiv \gamma i_\phi,\ \gamma^3$; B2: $(f\phi^2)^8 = (f\phi^2)^8$;

B3: $\{(f\phi)^2 f\}^8(8,4,8;2,6,2) = \{(ff)^6\phi\}^4 = (k\phi)^4$;

$$k \equiv H_\phi,\ \gamma\cdot\phi,\ (f\phi)^4 \equiv (1),\ (2),\ (3);$$

B31: $(\phi H_\phi)^4 = j_\phi$; B32: $\gamma(\phi\phi)^4 = \gamma i_\phi$;

B33: $\{\phi f)^4\phi\}^4(4,8,4;4,0,4) = (\phi^2 f)^8$.

So that if we take the forms γ, i_ϕ, j_ϕ, $(\phi^2 f)^8$ as fundamental, we have $B = (fi)^8$ reduced in terms of $\gamma\cdot i_\phi$, γ^3, $(\phi^2 f)^8$, j_ϕ.

The next invariant is of fourth degree, $C = (kk)^4$. From Syzygy (II), k is given in terms of $\gamma\cdot\phi$, H_ϕ, $(f\phi)^4$ [(1), (2), (3)]. So we take the fourth transvectant of k with each of its three members.

C1: $\gamma(\phi k)^4 \equiv \gamma j_\phi,\ \gamma^2 i_\phi,\ (\phi^2 f)^8$.

C2: $(kH_\phi)^4$; $k \equiv H_\phi,\ \gamma\cdot\phi,\ (f\phi)^4 \equiv (1),\ (2),\ (3)$.

C21: $(H_\phi H_\phi)^4 = \frac{1}{6} i_\phi^2$; C22: $\gamma(\phi H_\phi)^4 = \gamma j_\phi$;

C23: $\{(\phi f)^4 H_\phi\}^4(4,8,4;4,0,4) = (\phi H_\phi, f)^8$.

C3: $\{(f\phi)^4 k\}^4$; $k \equiv H_\phi,\ \gamma\cdot\phi,\ (f\phi)^4 \equiv (1),\ (2),\ (3)$.

C31: $\{(\phi f)^4 H_\phi\}^4 = (\phi H_\phi, f)^8$; C32: $\gamma\{\phi(\phi f)^4\}^4 = \gamma(\phi^2 f)^8$;

C33: $\{(f\phi)^4(f\phi)^4\}^4(8,4,4;0,4,4) = \{(f(f\phi)^4)^4\phi\}^4$;

$\{(f\phi)^4 f\}^4(8,4,8;0,4,4) = (i\phi)^4 + (12/7)(k\phi)^2 + (1/5)A\cdot\phi.$

$\equiv (1), \qquad (2), \qquad (3).$

C331: $\{(\phi i)^4\phi\}^4 = (\phi^2 i)^8$; $i \equiv \phi^2, \gamma f, (f\phi)^2 \equiv (a), (b), (c).$

C331a: $(\phi^2\phi^2)^8 = (\phi\cdot\phi, \phi^2)^8 = \{(\phi\phi^2)^4\phi\}^4 = (18/35)i_\phi{}^2$; *

C331b: $(\phi^2, \gamma f)^8 = \gamma(\phi^2, f)^8$;

C331c: $\{(f\phi)^2\phi^2\}^8 = \{(\phi\phi^2)^2 f\}^8 = (5/7)(\phi H_\phi, f)^8.$

C332: $\{(\phi k)^2\phi\}^4 = (H_\phi k)^4 \equiv i_\phi{}^2, \gamma j_\phi, (\phi H_\phi, f)^8$; [Cf. C2].

C333: $A(\phi\phi)^4 = Ai_\phi \equiv \gamma^2 i_\phi, i_\phi{}^2.$

So the invariant $C = (kk)^4$ is reducible as $\gamma^2\cdot i_\phi, \gamma j_\phi, i_\phi{}^2, \gamma(\phi^2 f)^8, (\phi H_\phi, f)^8$.

The next following invariant is the one of fifth degree

$$f_{kk} = (f_k k)^4 = \{(fk)^4 k\}^4.$$

In order to accomplish the reduction here we must first secure the expression for the covariant form f_k:

$f_k = (fk)^4$; $k \equiv \gamma\cdot\phi, H_\phi, (f\phi)^4 \equiv (1), (2), (3).$

$f_k 1$: $(f, \gamma\phi)^4 = \gamma(f\phi)^4$; $f_k 2$: $(fH_\phi)^4 = (fH_\phi)^4$;

$f_k 3$: $\{(f\phi)^4 f\}^4 = (i\phi)^4 + (12/7)(k\phi^2) + (1/5)A\cdot\phi \equiv (1), (2), (3).$

31: $(i\phi)^4$; $i \equiv (f\phi)^2, \gamma f, \phi^2 \equiv (1), (2), (3).$

311: $\{(f\phi)^2\phi\}^4 = \{(\phi f)^2\phi\}^4(4,8,4;2,2,2)$;

$$\sum_{i=0}^{2} a_i\{(\phi f)^{2+i}\phi\}^{4-i} = \sum_{j=0}^{0} b_j\{(\phi\phi)^{2+j}, f\}^{4-j}$$

where the actual numerical values of the literal coefficients a_i, b_j in these summations can be written down from the general form for Gordan's series given above.

$\{(\phi f)^2\phi\}^4 = -\{(\phi f)^3\phi\}^3 - (2/7)\{(\phi f)^4\phi\}^2 + (H_\phi f)^4;$

$\{(\phi f)^3\phi\}^3(4,8,4;2,1,3)$; $\sum_{i=0}^{1} a_i\{(\phi f)^{3+i}\phi\}^{3-i} = \sum_{j=0}^{1} b_j\{(\phi\phi)^{1+j} f\}^{5-j}$

$\{(\phi f)^3\phi\}^3 = -(1/2)\{(\phi f)^4\phi\}^2 + (1/2)(H_\phi f)^4$

$\{(\phi f)^4\phi\}^2(4,8,4;2,0,4)$; $\{(\phi f)^4\phi\}^2 = \sum_{j=0}^{2} b_j\{(\phi\phi)^j f\}^{6-j}$

* Grace and Young, p. 92.

$\{(\phi f)^4\phi\}^2 = (\phi^2 f)^6 + (2/7)(H_\phi f)^4$

$\therefore \{(\phi f)^2\phi\}^4 = (3/14)(\phi^2 f)^6 + (55/98)(H_\phi f)^4$

312: $(\phi, \gamma f)^4 = \gamma(f\phi)^4$; 313: $(\phi^2\phi)^4 = (18/35) i_\phi \cdot \phi$.

32: $(k\phi)^2$; $k \equiv (f\phi)^4, H_\phi, \gamma\cdot\phi \equiv (1), (2), (3)$.

321: $\{(\phi f)^4\phi\}^2 = (\phi^2 f)^6 + (2/7)(H_\phi f)^4$; 322: $(H_\phi\phi)^2 = (1/6) i_\phi\phi$;

323: $\gamma(\phi\phi)^2 = \gamma H_\phi$; 33: $A\cdot\phi \equiv \gamma^2\phi, i_\phi\cdot\phi$.

Hence f_k is expressed in terms of

$$\gamma(f\phi)^4,\ (H_\phi f)^4,\ (\phi^2 f)^6,\ i_\phi\phi,\ \gamma H_\phi,\ \gamma^2\phi.$$

In the work to follow there is no reason why we should retain the numerical coefficients which occur in the expansion by Gordan's series, and in fact our work will be very much easier if we do not. Then we can make use of a short cut which will be of very great advantage in saving time and labor. For instance, in reducing the form $\{(f\phi)^2\phi\}^4$, we notice that it is expressed in terms of $\{(f\phi)^3\phi\}^3$ and $\{(f\phi)^4\phi\}^2$. Furthermore when we have reduced these and substituted back, we notice that all the different types of terms which occur in $\{(f\phi)^2\phi\}^4$ also occur in $\{(f\phi)^4\phi\}^2$. We are interested only in the different types of terms which can arise, and so it is much easier to use $\{(f\phi)^4\phi\}^2$ than $\{(f\phi)^2\phi\}^4$ and so avoid the troublesome reduction on the left side of the series and the substitution. For a given form it is always possible to tell by inspection the form by which it can be replaced, but the method must be used with caution not to violate the conditions under which Gordan's series may be used. This device is used continuously in the work to follow.

We are now prepared to reduce the invariant f_{kk} of the fifth degree.

$f_k \equiv \gamma(f\phi)^4, (fH_\phi)^4, i_\phi\cdot\phi, \gamma H_\phi, \gamma^2\cdot\phi, (\phi^2 f)^6 \equiv (1), (2), (3), (4), (5), (6)$.

$k \equiv (f\phi)^4, H_\phi, \gamma\cdot\phi \equiv (1), (2), (3)$.

$f_{kk}11$: $\gamma\{(f\phi)^4(f\phi)^4\}^4 \equiv \gamma^3 i_\phi, \gamma^2 j_\phi, \gamma\cdot i_\phi^2, \gamma^2(\phi^2 f)^8, \gamma(\phi H_\phi, f)^8$;

12: $\gamma\{(\phi f)^4 H_\phi\}^4 = \gamma(\phi H_\phi, f)^8$; 13: $\gamma\{(\phi f)^4, \gamma\phi\}^4 = \gamma^2(\phi^2 f)^8$.

21: $\{(fH_\phi)^4(f\phi)^4\}^4 = \{(f(f\phi)^4)^4 H_\phi\}^4$;

$\{f(f\phi)^4\}^4 \equiv (\phi^2 f)^6, (H_\phi f)^4, \gamma(f\phi)^4, i_\phi\cdot\phi, \gamma H_\phi, \gamma^2\cdot\phi$

$\equiv (1), (2), (3), (4), (5), (6)$.

211: $\{(\phi^2 f)^6 H_\phi\}^4 = \{\phi^2 H_\phi)^2 f\}^8 \equiv (H_\phi^2 f)^8, i_\phi(\phi^2 f)^8$;

212: $\{(H_\phi f)^4 H_\phi\}^4 = (H_\phi^2 f)^8$; 213: $\gamma\{(\phi f)^4 H_\phi\}^4 = \gamma(\phi H_\phi, f)^8$;

214: $i_\phi(\phi H_\phi)^4 = i_\phi j_\phi$; 215: $\gamma(H_\phi H_\phi)^4 = \frac{1}{6}\gamma i_\phi^2$.

31: $i_\phi\{\phi(f\phi)^4\}^4 = i_\phi(\phi^2 f)^8$; 32: $i_\phi(\phi H_\phi)^4 = i_\phi j_\phi$; 33: $\{i_\phi\phi, \gamma\phi\}^4 = \gamma i_\phi^2$.

41: $\gamma\{H_\phi(f\phi)^4\}^4 = \gamma(\phi H_\phi, f)^8$; 42: $\gamma(H_\phi H_\phi)^4 = \frac{1}{6}\gamma i_\phi^2$;

43: $\{\gamma H_\phi, \gamma\phi\}^4 = \gamma^2 j_\phi$.

51: $\gamma^2\{\phi(\phi f)^4\}^4 = \gamma^2(\phi^2 f)^8$; 52: $\gamma^2(\phi H_\phi)^4 = \gamma^2 j_\phi$;

53: $\{\gamma^2\phi, \gamma\phi\}^4 = \gamma^3 i_\phi$.

61: $\{(\phi^2 f)^6(f\phi)^4\}^4 = \{(f(f\phi)^4)^2\phi^2\}^8$.

$\{(f\phi)^4 f\}^2\ (8, 4, 8; 0, 2, 4) \equiv (H\phi)^4,\ (i\phi)^2,\ k\cdot\phi \equiv (1),\ (2),\ (3)$.

611: $\{(\phi H)^4\phi^2\}^8 = (\phi^3, H)^{12}$;

612: $\{(\phi i)^2\phi^2\}^8 = \{(\phi\phi^2)^2 i\}^8 \equiv (\phi H_\phi, i)^8$;

$i \equiv (f\phi)^2,\ \gamma f,\ \phi^2 \equiv (a),\ (b),\ (c)$.

612a: $\{(\phi f)^2, \phi H_\phi\}^8 = \{(\phi, \phi H_\phi)^2 f\}^8 \equiv (H_\phi{}^2 f)^8,\ i_\phi(\phi^2 f)^8$;

612b: $\{\gamma f, \phi H_\phi\}^8 = \gamma(f, \phi H_\phi)^8$; 612c. $(\phi^2, \phi H_\phi)^8 \equiv i_\phi j_\phi$.

613: $(\phi k, \phi^2)^8 = \{(\phi\phi^2)^4 k\}^4 \equiv i_\phi(\phi k)^4 \equiv i_\phi j_\phi,\ \gamma i_\phi^2, i_\phi(\phi^2 f)^8$;

62: $\{(\phi^2 f)^6 H_\phi\}^4 = \{(\phi^2 H_\phi)^2 f\}^8 \equiv (H_\phi{}^2 f)^8,\ i_\phi(\phi^2 f)^8$;

63: $\{(\phi^2 f),^6 \gamma\phi\}^4 = \gamma\{(\phi^2\phi)^2 f\}^8 = \gamma(\phi H_\phi, f)^8$.

Conclusion: f_{kk} is reduced rationally and integrally in terms of:

$$\gamma i_\phi^2,\ \gamma^2 j_\phi,\ \gamma^3 i_\phi,\ \gamma(\phi H_\phi, f)^8,\ \gamma^2(\phi^2 f)^8,\ i_\phi(\phi^2 f)^8,\ (H_\phi{}^2 f)^8,\ i_\phi j_\phi,\ (\phi^3 H)^{12}.$$

So far we have as basic invariants γ, i_ϕ, j_ϕ, $(\phi^2 f)^8$, $(\phi H_\phi, f)^8$, $(H_\phi{}^2 f)^8$, $(\phi^3 H)^{12}$. The last two have occurred first in f_{kk}.

We are interested next in the invariant of sixth degree,

$$i_{kk} = (i_k k)^4 = \{(ik)^4 k\}^4.$$

To reduce it we must first get the forms appearing in the quartic covariant i_k. From the first two syzygies (I) and (II) we have

$$i \equiv (f\phi)^2,\ \gamma f,\ \phi^2; \qquad k \equiv (f\phi)^4,\ H_\phi,\ \gamma\cdot\phi.$$

We find that i_k is reduced in terms of $(t_\phi f)^5$, $i_\phi(\phi f)^4$, $\gamma(H_\phi f)^4$, $i_\phi H_\phi$, $j_\phi\phi$, $(\phi^2 f)^8\cdot\phi$, $(\phi H_\phi, f)^6$, $\gamma i_\phi\phi$, $\gamma^2 H_\phi$, $(H\phi^2)^8$, $\gamma(\phi^2 f)^6$, $\gamma^2(f\phi)^4$, $\gamma^3\phi$.

i_k is a covariant form of order four, degree four.

Then taking the fourth transvectant of each term of i_k with each one of the three forms of k, it can be shown that i_{kk} can be expressed in terms of the following:

$j_\phi(\phi^2 f)^8$, $i_\phi(\phi H_\phi, f)^8$, $\gamma(H_\phi{}^2 f)^8$, $\gamma i_\phi(\phi^2 f)^8$, $i_\phi{}^3$, $j_\phi{}^2, \gamma i_\phi j_\phi$, $\gamma^2 i_\phi{}^2$, $\gamma^2(\phi H_\phi, f)^8$, $\gamma^3 j_\phi$, $j_\phi(\phi^2 f)^8$, $(\phi^2 f)^8 \cdot (\phi^2 f)^8$, $(\phi^2 H_\phi, H)^{12}$, $\gamma(\phi^3 H)^{12}$, $\gamma^3(\phi^2 f)^8$, $\gamma^4 i_\phi$.

The fundamental invariants are:

$$\gamma,\ i_\phi,\ j_\phi,\ (\phi^2 f)^8,\ (\phi H_\phi, f)^8,\ (H_\phi{}^2 f)^8,\ (\phi^3 H)^{12},\ (\phi^2 H_\phi, H)^{12}.$$

The invariant $f_{k\Delta} = (f_k \Delta)^4$ comes next in order. To reduce it we need first the covariant $\Delta = (kk)^2$. We find that Δ may be replaced by:

$$(H\phi^2)^8,\ (\phi^2 f)^8 \cdot \phi,\ i_\phi(\phi f)^4,\ (t_\phi f)^5,\ \gamma(H_\phi f)^4,\ i_\phi H_\phi,\ j_\phi\phi,\ \gamma i_\phi\phi,\ (\phi H_\phi, f)^6,\ \gamma(\phi^2 f)^6,\ \gamma^2 H_\phi.$$

At the same time we have

$$f_k \equiv \gamma(f\phi)^4,\ (fH_\phi)^4,\ i_\phi\phi,\ \gamma H_\phi,\ \gamma^2\phi,\ (\phi^2 f)^6.$$

We take the fourth transvectant of f_k with Δ. The result is that $f_{k\Delta}$ is reduced in terms of the following transvectants:

$\gamma(\phi^2 f)^8(\phi^2 f)^8$, $\gamma i_\phi{}^3$, $\gamma i_\phi(\phi H_\phi, f)^8$, $\gamma j_\phi{}^2$, $\gamma j_\phi(\phi^2 f)^8$, $\gamma(\phi^2 H_\phi, H)^{12}$, $\gamma^2 i_\phi(\phi^2 f)^8$, $\gamma^2 i_\phi j_\phi$, $\gamma^2(H_\phi{}^2 f)^8$, $\gamma^2(\phi^3 H)^{12}$, $\gamma^2 i_\phi j_\phi$, $\gamma^3 i_\phi{}^2$, $\gamma^3(\phi H_\phi, f)^8$, $\gamma^4 j_\phi$, $(\phi^2 f)^8(\phi H_\phi, f)^8$, $i_\phi(H_\phi{}^2 f)^8$, $i_\phi(\phi^3 H)^{12}$, $i_\phi{}^2(\phi^2 f)^8$, $i_\phi{}^2 j_\phi$, $j_\phi(\phi H_\phi, f)^8$, $(\phi H_\phi{}^2, H)^{12}$.

Then we have found as fundamental invariants thus far:

$$\gamma,\ i_\phi,\ j_\phi,\ (\phi^2 f)^8,\ (\phi H_\phi, f)^8,\ (H_\phi{}^2 f)^8,\ (\phi^3 H)^{12},\ (\phi^2 H_\phi, H)^{12}\ (\phi H_\phi{}^2, H)^{12}.$$

The invariant next to be considered is $i_{k\Delta} = (i_k \Delta)^4$ of eighth degree. i_k is expressed by means of:

$$i_\phi(f\phi)^4,\ \gamma^2(f\phi)^4,\ \gamma(fH_\phi)^4,\ j_\phi\phi,\ (\phi^2 f)^8 \cdot \phi,\ \gamma i_\phi\phi,\ \gamma^3\phi,$$
$$i_\phi H_\phi,\quad \gamma^2 H_\phi,\quad \gamma(\phi^2 f)^6,\quad (t_\phi f)^5,\quad (\phi H_\phi, f)^6,\quad (H\phi^2)^8.$$

Δ is reduced in terms of:

$$(H\phi^2)^8,\ (\phi^2 f)^8\phi,\ i_\phi(f\phi)^4,\ (t_\phi f)^5,\ \gamma(H_\phi f)^4,\ i_\phi H_\phi,\ j_\phi\phi,$$
$$\gamma i_\phi\phi,\quad (\phi H_\phi, f)^6,\ \gamma(\phi^2 f)^6,\ \gamma^2 H_\phi.$$

Now if we take the fourth transvectant of all members in one set with all those in the other we can show that $i_{k\Delta}$ is expressed in terms of the following invariants.

$\gamma i_\phi(\phi^3H)^{12}$, $\gamma i_\phi(H^2\phi f)^8$, $\gamma i_\phi{}^2 j_\phi$, $\gamma i_\phi{}^2(\phi^2f)^8$, $\gamma j_\phi(\phi H_\phi, f)^8$, $\gamma(\phi^2f)^8(\phi H_\phi, f)^8$,
$\gamma(\phi H_\phi{}^2, H)^{12}$, $\gamma^2 i_\phi{}^3$, $\gamma^2 i_\phi(\phi H_\phi, f)^8$, $\gamma^2 j_\phi(\phi^2f)^8$, $\gamma^2 j_\phi{}^2$, $\gamma^2(\phi^2f)^8(\phi^2f)^8$,
$\gamma^2(\phi^2H_\phi, H)^{12}$, $\gamma^3 i_\phi j_\phi$, $\gamma^3 i_\phi(\phi^2f)^8$, $\gamma^3(\phi^3H)^{12}$, $\gamma^3(H_\phi{}^2f)^8$, $\gamma^4 i_\phi{}^2$,
$\gamma^4(\phi H_\phi, f)^8$, $\gamma^5 j_\phi$, $i_\phi{}^4$, $i_\phi{}^2(\phi H_\phi, f)^8$, $i_\phi j_\phi{}^2$, $i_\phi j_\phi(\phi^2f)^8$,
$i_\phi(\phi^2f)^8(\phi^2f)^8$, $i_\phi(\phi^2H_\phi, H)^{12}$, $j_\phi(\phi^3H)^{12}$, $j_\phi(H_\phi{}^2f)^8$, $(\phi^2f)^8(\phi^3H)^{12}$, $(\phi^2f)^8(H_\phi{}^2f)^8$,
$(\phi H_\phi, f)^8(\phi H_\phi, f)^8$, $(H_\phi{}^3H)^{12}$.

The basic invariants are

$$\gamma,\ i_\phi,\ j_\phi,\ (\phi^2f)^8,\ (\phi H_\phi, f)^8,\ (H_\phi{}^2f)^8,\ (\phi^3H)^{12},$$
$$(\phi^2H_\phi, H)^{12},\ (\phi H_\phi{}^2, H)^{12},\ (H_\phi{}^3H)^{12}.$$

We consider in sequence the invariant of ninth degree, $f_{\Delta\Delta} = (f_\Delta\Delta)^4$. Hence we must first reduce the quartic covariant f_Δ.

Δ is reduced in terms of the following covariant forms:

$(H\phi^2)^8$, $(\phi^2f)^8\phi$, $i_\phi(\phi f)^4$, $(t_\phi f)^5$, $\gamma(H_\phi f)^4$, $i_\phi H_\phi$, $j_\phi\phi$, $\gamma i_\phi\phi$,
$(\phi H_\phi, f)^6$, $\gamma(\phi^2f)^6$, $\gamma^2 H_\phi$.

From these we have f_Δ given by

$i_\phi{}^2\phi$, $\gamma j_\phi\phi$, $\gamma^2 i_\phi\phi$, $\gamma(\phi^2f)^8\phi$, $(\phi H_\phi, f)^8\phi$
$\gamma i_\phi H_\phi$, $\gamma^3 H_\phi$, $j_\phi H_\phi$, $(\phi^2f)^8 H_\phi$, $(\phi^2f)^8(\phi f)^4$,
$\gamma i_\phi(\phi f)^4$, $j_\phi(\phi f)^4$, $i_\phi(H_\phi f)^4$, $\gamma^2(H_\phi f)^4$, $\gamma^2(\phi^2f)^6$,
$i_\phi(\phi^2f)^6$, $\gamma(\phi^2H)^8$, $(\phi H_\phi, H)^8$, $\gamma(iH_\phi)^4$, $(it_\phi)^5$,
$\gamma(i\phi^2)^6$, $(\phi H_\phi, i)^6$, $\gamma\,(kH_\phi)^2$, $i_\phi(k\phi)^2$, $(kt_\phi)^3$,
$(\phi^2p)^8$, $(\phi^2f_1)^7$, $(\phi^2f_2)^6$, $(\phi^2f_3)^5$, $(H_\phi f_k)^2$,
$i_\phi f_k$.

Then using this set and the set for Δ just preceding, we start to reduce $f_{\Delta\Delta}$. This is a very long and involved process, and due to the style employed, no record has been kept of all the possible combinations of factors which occur. It can be shown definitely however that $f_{\Delta\Delta}$ is composed of invariant terms, every one of which is a product of invariants taken from the group of fundamental forms following:

$\gamma,\ i_\phi,\ j_\phi,\ (\phi^2f)^8, (\phi H_\phi, f)^8, (H_\phi{}^2f)^8,\ (\phi^3H)^{12},\ (\phi^2H_\phi, H)^{12},\ (\phi H_\phi{}^2, H)^{12},$
$(H_\phi{}^3H)^{12},\ (\phi^3t_\phi, T)^{18}.$

The last form is new and occurs for the first time in this ninth degree invariant.

The last invariant is $i_{\Delta\Delta}$ of tenth degree. As above, we need the reduced form of the intermediate covariant form i_Δ, of degree six and order four. We remember that i is given by $(f\phi)^2$, γf, ϕ^2 and that

$$\Delta \equiv (H\phi^2)^8,\ (\phi^2 f)^8 \cdot \phi,\ i_\phi(\phi f)^4,\ (t_\phi f)^5,\ \gamma(H_\phi f)^4,\ i_\phi H_\phi,$$
$$j_\phi\phi,\ \gamma i_\phi\phi,\ (\phi H_\phi, f)^6,\ \gamma(\phi^2 f)^6,\ \gamma^2 H_\phi.$$

By taking the cross-transvectants we find that set of forms which we may use for i_Δ. In the reduction followed out there are ninety-one of these forms, and a list of them will not be given here. This does not represent the minimum number of such forms however, for it is often hard to tell which of several covariant forms is the simplest. It is sufficient to say that part of the forms in this list of ninety-one are reducible in terms of the others. But it appears easier to reduce them in a transvectant which is an invariant than before this point. From this point of view we use methods in reducing invariants which would be quite long and involved in the reduction of covariants.

From the list of forms comprising i_Δ and the constituent forms of Δ itself given previously we are ready to undertake the task of reducing the tenth degree invariant. The procedure is so long and tedious that we shall merely state the results. Most of the reductions are accomplished in terms of intermediate forms which have already been treated.

We find that $i_{\Delta\Delta}$ is composed of members which are compounded of the list of basic invariants following:

$$\gamma,\ i_\phi,\ j_\phi,\ (\phi^2 f)^8,\ (\phi H_\phi, f)^8,\ (H_\phi{}^2 f)^8,\ (\phi^3 H)^{12},\ (\phi^2 H_\phi, H)^{12},\ (\phi H_\phi{}^2, H)^{12},$$
$$(H_\phi{}^3 H)^{12},\ (\phi^3 t_\phi, T)^{18},\ (\phi^2 H_\phi t_\phi, T)^{18}.$$

Now it has been shown that the nine invariants of the Jacobian octavic f, of degree two to ten inclusive, can be replaced by the twelve fundamental invariants given above. In the higher degree invariants the reduction has not actually been carried so far, but has been accomplished in terms of intermediate forms which have already been shown to be equivalent to these twelve forms.

The twelve fundamental forms found are characterized, aside from the first three, by the fact that the transvectant symbol representing them has only forms of the quartic on one side of the symbol, forms of the octavic on the other. In this sense they might be called pure. The invariant γ is a joint invariant of Q and Q' and does not occur in the simultaneous system

of f and ϕ. The two invariants i_ϕ and j_ϕ are basic invariants of the quartic ϕ. The others are joint transvectants of the simpler forms of f and ϕ.

It remains for us to reconcile our results with the general theory on which the investigation is based and in this respect the work is incomplete. Let us refer again to the three relations of Stroh, connecting f, ϕ and γ with the covariants of the apolar octavic $F = F_x^8$. We note that f, ϕ, γ, each multiplied by the power of the resultant R, are equated to expressions involving covariant forms of F, and that each one of these three expressions is of degree sixteen in the p's. All our twelve fundamental forms are transvectants of the three forms f, ϕ, γ and hence if we perform the transvection on the right as indicated we can ultimately reduce these to polynomials in the invariants $I_2 \cdots I_{10}$ of F, ranging in degree from eight to forty, for $I_2 \cdots I_{10}$ comprise the complete system of invariants of F.

Theoretically we should get nine invariants for the complete system and we have twelve. From the form of Stroh's three relations we may infer that the resultant R might be added to the list and that three forms of the list are superfluous. But R itself can finally be expressed in terms of these twelve forms, from the form for R given by Gordan on p. 389 of Vol. 3 of the *Mathematische Annalen*. While we are able to get all our twelve forms expressed as polynomials in $I_2 \cdots I_{10}$ of the apolar octavic, we have no obvious way of getting the $I_2 \cdots I_{10}$ in terms of γ, i_ϕ, j_ϕ, $\cdots$, etc., and this prevents us from drawing conclusions as to the number of independent forms in this system.

If we examine our list of twelve fundamental forms more closely we are led to several additional conclusions. In the first place, the invariant γ which is certainly a combinant form of Q and Q' is not found in the joint system of covariants of f and ϕ, for it is of the first degree while a transvectant from this system must be at least of the second degree. Next, while we find the transvectant forms $(\phi^3 t_\phi, T)^{18}$, $(\phi^2 H_\phi t_\phi, T)^{18}$ in our set, we do not meet the remaining two forms of similar composition, $(\phi H_\phi^2 t_\phi, T)^{18}$, $(H_\phi^3 t_\phi, T)^{18}$, as we might expect from symmetry. This might be advanced as an intuitional argument for believing that the first two mentioned are superfluous. Furthermore no additional forms of this type can occur, for T is the highest ordered covariant in the system of f which can be used, and if we had on the right side of our transvectant symbol the product of two forms of f for a transvectant of higher index, the form would be reducible by a criterion due to Gordan.

So we are still faced with the problem of determining how many of these

forms furnish a complete system of invariant combinants for our two quintics. This question even for the lower ordered forms has not been treated exhaustively in the literature. For instance the treatment of the case of two quartics by Rowe * is not complete. It is hoped that some means of treating this problem in general can be secured. And of course we are still faced with the problem, much larger in scope, of the complete system of covariant combinants of the two quintics, of which the matter of invariants is only a small part.

* J. E. Rowe, *Transactions of the American Mathematical Society*, Vol. 12, p. 305.

A Boundary Value Problem of Ordinary Self-Adjoint Differential Equations with Singularities.

By Marion Cameron Gray.

Introduction. The problem of solving the ordinary homogeneous self-adjoint differential equation of the second order with singularities, subject to given boundary conditions, has been discussed by Hilb.* He considers the non-singular equation

$$(d/dx)(xdu/dx) + [-g(x) + \lambda h(x)/x]\, u = 0 \qquad (\epsilon \leq x \leq 1)$$

with the boundary conditions $u(\epsilon) = u(1) = 0$, and introduces a singularity by allowing ϵ to become zero. The functions g and h are continuous, and h is positive in $(\epsilon, 1)$. Hilb finds a Green's function for the equation, and the corresponding integral equation; then, in passing to the limiting case $\epsilon = 0$, he applies the Hilbert † theory of quadratic forms in infinitely many variables.

Weyl ‡ has discussed the more general equation

$$(d/dx)(p(x)du/dx) - q(x)u + \lambda u = 0 \qquad (0 \leq x < \infty)$$

with homogeneous boundary conditions at zero and infinity. The function q is continuous, and p is continuous and positive in the interval $(0, \infty)$, so that the only singularity is at infinity. Weyl also introduces a Green's function, and applies to the resulting integral equation his theory of singular integral equations § which is based on Hilbert's work. He distinguishes two types of

* E. Hilb, "Über Integraldarstellungen willkürlicher Funktionen," *Mathematische Annalen*, Vol. 66, pp. 1-66. The work of Wirtinger, *Mathematische Annalen*, Vol. 48, pp. 387-389, may also be cited.

† D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig (1912).

‡ H. Weyl, "Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen," *Göttinger Nachrichten*, Mathematische-Physische Klasse, 1909, pp. 37-63, also "Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen," *Mathematische Annalen*, Vol. 68, pp. 220-269.

§ H. Weyl, "Singuläre Integralgleichungen," *Mathematische Annalen*, Vol. 66, pp. 273-324.

singularity, the limit circle type, which leads to solutions for complex values of λ, and the limit point type, which leads to solutions only for real values of λ.

In a later paper * Weyl extends his method to the equation

$$(d/dx)(p(x)du/dx) - q(x)u + \lambda k(x)u = 0 \qquad (-\infty < x < \infty)$$

where the function k is continuous, and there are singularities at both ends of the interval $(-\infty, \infty)$, first for the case in which k is positive, and then for the polar case in which k may change sign.

In the present paper we consider the equation

$$(d/dx)(p(x)du/dx) + q(x)u + \lambda u = 0 \qquad (0 \leq x \leq \pi)$$

with boundary conditions $u(0) = u(\pi) = 0$. We assume that the function q is continuous, and that the function p is continuous with a continuous first derivative, and vanishes a finite number of times in the interval. We introduce a complete normed orthogonal system of functions, $\phi_m(x)$, satisfying the boundary conditions, and use it to pass from the differential equation to a system of infinitely many equations in an infinite number of unknowns.† We consider the case in which this system of equations has solutions only for real values of λ, Weyl's limit point type, and we apply the results of the Carleman ‡ theory of integral equations of Class I, for the special case of quadratic forms. From the solutions of the system of infinitely many equations we derive solutions of the given differential equation and discuss their properties, referring in this connexion also to the Hellinger § theory of limited quadratic forms (§§ 1 and 2).

In § 3 we introduce a theory of *limited differential systems,* by means of which we derive (§ 4) an expansion theorem for an arbitrary function $f(x)$, subject to the conditions that f be continuous, satisfy the boundary conditions, and have a first derivative which is integrable and of integrable square in $(0, \pi)$. The corresponding expansion theorems of Weyl and Hilb require further the existence of the function

---

* H. Weyl, "Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen," *Göttinger Nachrichten*, Mathematische-Physische Klasse, 1910, pp. 442-467. For further references to the literature of the subject we may refer to those given by Weyl in the introduction to this paper.

† Anna Pell Wheeler, "Linear Ordinary Self-Adjoint Differential Equations of the Second Order," *American Journal of Mathematics*, Vol. 49 (1927), p. 310.

‡ T. Carleman, *Sur les équations intégrales singulières à noyau reél et symétrique.*

§ E. Hellinger, "Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen," *Journal für Mathematik*, Vol. 136, pp. 211-271.

$$(d/dx)(p(x)df/dx),$$

though we may note that Plancherel * has obtained an expansion theorem for the Hilb problem in which he requires only that f be continuous and of bounded variation in $(0, 1)$, vanish at each end of the interval, and be such that the integral

$$\int_0^1 |f(t)/t|\, dt$$

converges.

1. *Properties of the Solutions of the System of Infinitely Many Equations Arising from the Differential Equation.* We consider solutions of the linear homogeneous self-adjoint differential equation of the second order

$$(d/dx)(p(x)du/dx) + q(x)u + \lambda u = 0 \tag{1}$$

which are of integrable square in the interval $(0, \pi)$. The function q is continuous in $(0, \pi)$, the function p is continuous with a continuous first derivative, and vanishes a finite number of times in the interval. The boundary conditions are assumed to be

$$u(0) = u(\pi) = 0, \tag{2}$$

if $p(0) \neq 0$ and $p(\pi) \neq 0$, but if for example $p(0) = 0$ the conditions are

$$\lim_{x \to 0} p(x)u(x) = 0 \text{ and } u(\pi) = 0. \tag{2'}$$

An auxiliary system of normed orthogonal functions satisfying the boundary conditions (2) is

$$\phi_m(x) = (2/\pi)^{1/2} \sin mx, \tag{3}$$

and this system is also complete.

To obtain from the system (1) and (2) [or (2')] a system of infinitely many equations in an infinite number of unknowns we multiply equation (1) by $\phi_m(x)$, and integrate between 0 and π,†

$$\int_0^\pi (pu')'\phi_m + \int_0^\pi qu\phi_m + \lambda \int_0^\pi u\phi_m = 0.$$

* M. Plancherel, "Integraldarstellungen willkürlicher Funktionen," *Mathematische Annalen*, Vol. 67, pp. 519-534.

† When there is no ambiguity we shall omit the variable of integration; for definite integrals in $(0, \pi)$ this variable is always x. We use primes throughout to denote derivatives with respect to x.

Now

$$\int_0^\pi (pu')'\phi_m = \int_0^\pi (p\phi_m')'u,$$

since ϕ_m satisfies the boundary conditions (2), while u satisfies the conditions (2) or (2′). Also

$$\int_0^\pi (p\phi_m')'u = \sum_n \int_0^\pi (p\phi_m')'\phi_n \int_0^\pi u\phi_n$$

by the known properties of the complete system (3). A solution u of the system (1) and (2) [or (2′)] therefore satisfies the system of equations

$$\sum_{n=1}^{\infty} \int_0^\pi (p\phi_m')'\phi_n \int_0^\pi u\phi_n + \sum_{n=1}^{\infty} \int_0^\pi q\phi_m\phi_n \int_0^\pi u\phi_n + \lambda \int_0^\pi u\phi_m = 0,$$

which may be written

$$\sum_{n=1}^{\infty} p_{mn}x_n + \sum_{n=1}^{\infty} q_{mn}x_n + \lambda x_m = 0, \tag{4}$$

where

$$x_m = \int_0^\pi u\phi_m,$$

$$p_{mn} = \int_0^\pi (p\phi_m')'\phi_n,$$

and

$$q_{mn} = \int_0^\pi q\phi_m\phi_n.$$

The matrix $P = (p_{mn})$ is symmetric, the rows are of finite norm, that is Σp_{mn}^2 converges for every value of m, but P is not limited. The matrix $Q = (q_{mn})$ is symmetric and limited.

The matrix $P + Q$ is of the type for which Carleman has given an existence theorem.* In this paper we consider only the case in which $P + Q$ is of Class I,† that is, the system of equations (4) has no solution of finite norm, not identically zero, for complex values of λ.

The system of equations (4) may, however, have solutions of finite norm, not identically zero, for real values of λ, the characteristic values. Let these values be λ_a, and the corresponding characteristic solutions l_{am}. These characteristic values of λ form the *point spectrum*.

Let $\Delta f(\lambda) = f(\lambda_2) - f(\lambda_1)$ denote the increment of the continuous func-

* Carleman, *l. c.*, ch. 1.

† Carleman, *l. c.*, p. 187.

tion $f(\lambda)$ for the interval $\Delta = (\lambda_1, \lambda_2)$ of the real λ axis, and consider the system of equations

$$(5) \qquad \sum_n p_{mn}\,\Delta\rho_n(\lambda) + \sum_n q_{mn}\,\Delta\rho_n(\lambda) + \int_\Delta \lambda d\rho_n(\lambda) = 0,$$

which may have solutions of finite norm, and not constant, for certain intervals Δ of the real λ axis. These intervals Δ form the *continuous spectrum,* and the corresponding characteristic differential solutions $\rho_m(\lambda)$ are continuous and of bounded variation in λ. They are determined up to an additive constant, which we fix by the condition $\rho_m(0) = 0$ for every value of m. The sequence $\{\rho_m(\lambda)\}$ defines a continuous monotonic non-decreasing function of bounded variation in λ

$$(6) \qquad \begin{aligned} \rho(\lambda) &= \sum_m [\rho_m(\lambda)]^2 && \lambda > 0 \\ &= 0 && \lambda = 0 \\ &= -\sum_m [\rho_n(\lambda)]^2 && \lambda < 0 \end{aligned}$$

The function $\rho(\lambda)$ is called the basis function of the solutions $\rho_m(\lambda)$.

The term *spectrum* is used to denote the values of λ which belong to the point spectrum, the limiting points of the point spectrum, and the continuous spectrum, and we know by the theory of forms of Class I that such a spectrum exists, that is, there are solutions of finite norm either of the linear system of equations (4), or of the differential system (5) or of both. Further, since the matrix P is not limited, the spectrum may extend over any part of the real λ axis.

We may note that there may be more than one sequence of characteristic differential solutions satisfying all the conditions imposed, but for the present we shall assume that the continuous spectrum is simple, and return later (§ 5) to the more general case.

The sequences $\{l_{am}\}$ and $\{\rho_m(\lambda)\}$ form an orthogonal system, satisfying the relations

$$(7) \qquad \begin{aligned} \sum_m l_{am}\, l_{\beta m} = e_{\alpha\beta} &= 0 && \alpha \neq \beta \\ &= 1 && \alpha = \beta, \end{aligned}$$

$$(8) \qquad \sum_m \Delta_1\rho_m(\lambda)\ \Delta_2\rho_m(\lambda) = \Delta_{12}\rho(\lambda),$$

where Δ_1 and Δ_2 are any two intervals of the real λ axis in which there is a continuous spectrum, and Δ_{12} the interval common to the two intervals Δ_1 and Δ_2. Finally the two types of solutions are orthogonal

$$\sum_m l_{am}\,\Delta\rho_m(\lambda) = 0 \tag{9}$$

for every α, and every value of λ.*

If the system of equations

$$\sum_n (p_{mn} + q_{mn})x_n = 0 \tag{10}$$

has no solution of finite norm other than zero, the solutions l_{am} and $\rho_m(\lambda)$ form a *complete* system, satisfying the relation

$$\sum_a l_{am}l_{an} + \int_{-\infty}^{\infty} \frac{d\rho_m(\lambda)\,d\rho_n(\lambda)\dagger}{d\rho(\lambda)} = e_{mn}. \tag{11}$$

The proof is easily obtained from the following result which is a special case of a theorem given by Carleman.‡

Let the matrix of the system of equations (4) be of Class I, with characteristic numbers λ_a and corresponding characteristic solutions l_{am}. Let the continuous spectrum be simple with characteristic differential forms $\rho_m(\lambda)$ and basis function $\rho(\lambda)$. Write

$$Y_a = \sum_n l_{an}y_n, \qquad X_a = \sum_n l_{an}x_n$$

$$\Delta y(\lambda) = \sum_n \Delta\rho_n(\lambda)y_n, \qquad \Delta x(\lambda) = \sum_n \Delta\rho_n(\lambda)x_n,$$

where $\{x_n\}$ and $\{y_n\}$ are sequences of finite norm, and $\{x_n\}$ is such that $\sum x_m[\sum(p_{mn} + q_{mn})^2]^{\frac{1}{2}}$ converges. Then

$$-\sum_n (p_{mn} + q_{mn})y_n = \sum_a \lambda_a Y_a l_{am} + \int_{-\infty}^{\infty} \frac{\lambda\, d\rho_m(\lambda)\, dy(\lambda)}{d\rho(\lambda)}\,, \tag{12}$$

$$-\sum_{m,n} (p_{mn} + q_{mn})x_m y_n = \sum_a \lambda_a X_a Y_a + \int_{-\infty}^{\infty} \frac{\lambda\, dx(\lambda)\, dy(\lambda)}{d\rho(\lambda)}\,. \tag{13}$$

The series and integrals involved are absolutely convergent.

These expansions show at once that there is no sequence of finite norm, not identically zero, orthogonal to all the sequences $\{l_{am}\}$ and $\{\rho_m(\lambda)\}$. For any two sequences $\{x_m\}$ and $\{y_n\}$ of finite norm we obtain expansions

$$y_m = \sum_a l_{am} \sum_n y_n l_{an} + \int_{-\infty}^{\infty} \frac{d\rho_m(\lambda)\; d\sum_n y_n\rho_n(\lambda)}{d\rho(\lambda)} \tag{14}$$

---

* For proofs of these statements about the spectrum we may refer to Hellinger, *l. c.*, pp. 240 et seq., and Carleman, *l. c.*, ch. 3.

† Hellinger, *l. c.*, p. 237, defines this integral in a finite interval, and the definition is extended to an infinite interval by Weyl, *l. c.*, *Mathematische Annalen*, Vol. 66, p. 288.

‡ Carleman, *l. c.*, pp. 101-102.

$$(15) \qquad \sum_m x_m y_m = \sum_a \sum_n l_{am} x_m \sum_n l_{an} y_n + \int_{-\infty}^{\infty} \frac{d \sum_m x_m \rho_m(\lambda) \, d \sum_n y_n \rho_n(\lambda)^*}{d\rho(\lambda)}$$

and the relation (11) follows immediately.

2. *The Solutions of the Differential Equation.* From the solutions l_{am} and $\rho_m(\lambda)$ of the last section we can now derive solutions of the differential equation (1). Consider the sequence $\{\rho_m(\lambda)\}$ which is a solution of finite norm for the system of equations (5). By the Riesz-Fischer theorem

$$(16) \qquad \rho_m(\lambda) = \int_0^\pi u(x,\lambda) \phi_m(x) \, dx$$

for a function $u(x, \lambda)$ which is of integrable square in x in $(0, \pi)$. Further the sequence $\{\Sigma \, p_{mn} \rho_n(\lambda)\}$ is of finite norm. Now

$$\begin{aligned} \sum_n p_{mn} \rho_n(\lambda) &= \sum_n \int_0^\pi (p\phi_m')' \phi_n \int_0^\pi u \phi_n \\ &= \int_0^\pi (p\phi_m')' u \\ &= -(2/\pi)^{\frac{1}{2}} m^2 \int_0^\pi pu \sin mx + (2/\pi)^{\frac{1}{2}} m \int_0^\pi p' u \cos mx \\ &= (2/\pi)^{\frac{1}{2}} m \left[\int_0^\pi p' u \cos mx - m \int_0^\pi pu \sin mx \right] \end{aligned}$$

The function u is such that this is of finite norm, and therefore the factor in the bracket is of finite norm. But the sequence $\{\int_0^\pi p' u \cos mx\}$ is itself of finite norm, therefore the sequence $\{m \int_0^\pi pu \sin mx\}$ is likewise. It follows that the function pu has a derivative almost everywhere which is integrable and of integrable square in the interval $(0, \pi)$, and, since p has a derivative, u has a derivative almost everywhere except perhaps for the zeros of the function p. We have also $u(0, \lambda) = u(\pi, \lambda) = 0$, unless $p(0) = 0$ or $p(\pi) = 0$, and in all cases $\lim_{x \to 0} p(x) u(x, \lambda) = \lim_{x \to \pi} p(x) u(x, \lambda) = 0$.

Further, the expression

$$m \int_0^\pi p' u \cos mx - m^2 \int_0^\pi pu \sin mx$$

* Carleman, *l. c.*, p. 103.

is of finite norm, and

$$-m^2 \int_0^\pi pu \sin mx = -m \int_0^\pi (pu)' \cos mx.$$

Hence

$$m \int_0^\pi [p'u - (pu)'] \cos mx$$

is of finite norm, and therefore the function $p'u - (pu)'$ has a derivative almost everywhere which is integrable and of integrable square in the interval $(0, \pi)$. Except at the zeros of the function p this function may be written pu', so that at all points for which $p(x) \neq 0$ the functions u' and $(pu')'$ exist almost everywhere. Further we have

$$\int_0^\pi [u(x,\lambda)]^2 dx = \sum_m [\int_0^\pi u\phi_m]^2 = \rho(\lambda),$$

and therefore $\int_0^\pi [u(x,\lambda)]^2 dx$ converges to a continuous function of λ.

Now Weyl * has defined a *differential solution* $u(x,\lambda)$ of equation (1) as a real continuous function of x and of λ, for which the integral $\int_0^\pi [u(x,\lambda)]^2 dx$ exists as a continuous function of λ, and which satisfies the boundary conditions (2) or (2′) and the equation

$$(pu')' + qu + \int_0^\lambda \lambda \, du \dagger = 0. \tag{17}$$

By the properties we have already found of the function $u(x,\lambda)$ defined by equation (16), we need only show that it satisfies equation (17) in order to prove that it is a differential solution of equation (1). Return to the system of equations (5) satisfied by $\rho_m(\lambda)$, and write it in the form

$$\sum_n \int_0^\pi (p\phi_m')' \phi_n \Delta\rho_n(\lambda) + \sum_n \int_0^\pi q \, \phi_m\phi_n \, \Delta\rho_n(\lambda) + \int_\Delta \lambda d\rho_m(\lambda) = 0,$$

Substitute for $\Delta\rho_m(\lambda)$ from equation (16)

$$\int_0^\pi (p\phi_m')' \Delta u(x,\lambda) + \int_0^\pi q\phi_m \Delta u(x,\lambda) + \int_0^\pi \phi_m \int_\Delta \lambda \, du(x,\lambda) = 0.$$

---

* Weyl. *l. c.*, *Mathematische Annalen*, Vol. 68, p. 239.

† In the Stieltjes and Hellinger integrals used throughout the variable of integration is always λ.

Take Δ to be the interval $(0, \lambda)$, then $\Delta u(x, \lambda) = u(x, \lambda)$. Further

$$\int_0^\pi (p\phi_m')' u(x, \lambda) = \int_0^\pi [p'u - (pu)']' \phi_m$$

by the boundary properties of the functions $u(x, \lambda)$ and $\phi_m(x)$, and therefore finally

$$\int_0^\pi \phi_m \{[p'u - (pu)']' + qu + \int_0^\lambda \lambda du(x, \lambda)\} = 0. \tag{18}$$

At all points for which $p(x) \neq 0$ this may be written

$$(pu')' + qu + \int_0^\lambda \lambda du(x, \lambda) = 0,$$

so that the function $u(x, \lambda)$ is a differential solution of equation (1), except perhaps at the zeros of the function p.

If we consider the solutions l_{am} of the system of equations (4) we can show in a similar manner that the functions $u_a(x)$ defined by

$$l_{am} = \int_0^\pi u_a(x)\, \phi_m(x)\, dx \tag{19}$$

are solutions of equation (1) except perhaps at the zeros of the function p.* The functions $u_a(x)$ are such that the sequence $\{ m \int_0^\pi p u_a \phi_m \}$ is of finite norm, so that the function pu_a has a derivative almost everywhere which is integrable and of integrable square in the interval $(0, \pi)$.

The solutions $u(x, \lambda)$ and $u_a(x)$ have orthogonal properties similar to those of the systems $\rho_m(\lambda)$ and l_{am}. They are derived immediately from the corresponding relations (7), (8) and (9), and have the form

$$\int_0^\pi u_a(x)\, u_\beta(x)\, dx = e_{a\beta}, \tag{20}$$

$$\int_0^\pi \Delta_1 u(x, \lambda)\, \Delta_2 u(x, \lambda)\, dx = \Delta_{12} \rho(\lambda) \tag{21}$$

for any two intervals Δ_1 and Δ_2 of the real λ axis, and

$$\int_0^\pi u_a(x)\, \Delta u(x, \lambda)\, dx = 0 \tag{22}$$

for every value of α, and every interval Δ.

---

* These solutions have been obtained for the non-singular differential equation $u'' + qu + \lambda u = 0$ by Anna Pell Wheeler, *l. c.*, *American Journal of Mathematics*, Vol. 49 (1927), p. 312.

3. *Limited Differential Systems.* We now introduce the conception of limited differential systems, and prove some properties of these systems analogous to the known properties of limited matrices. These are used later in obtaining an expansion theorem for an arbitrary function $f(x)$, subject to suitably imposed conditions, in terms of the solutions $u_a(x)$ and $u(x, \lambda)$ of § 2. The functions $\rho_m(\lambda)$ and $\rho(\lambda)$ used throughout this section are the characteristic differential solutions and corresponding basis function of § 1.

Definition I. Let $f(\lambda)$ be a continuous function of λ in the infinite interval $(-\infty, \infty)$, and let $u(\lambda)$ be continuous and of bounded variation in that interval. Then the function $\int_\Delta f(\lambda)\,du(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$ if the Hellinger integral

$$\int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[du(\lambda)]^2}{d\rho(\lambda)}$$

converges.* If $\int_\Delta f(\lambda)\,du(\lambda)$ is integrable $H(\rho)$, and if, further, $\Delta g(\lambda)$ is any function integrable $H(\rho)$, then the integral

$$\int_{-\infty}^{\infty} f(\lambda) \frac{du(\lambda)\,dg(\lambda)}{d\rho(\lambda)}$$

is absolutely convergent.

Definition II. The system $\int_\Delta f(\lambda)\,du_m(\lambda)$ is a *differential system* in $(-\infty, \infty)$ if $\int_\Delta f(\lambda)\,du_m(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$ for every value of m, and if, further, the series $\sum [u_m(\lambda)]^2$ converges to a continuous function $u(\lambda)$.

Definition III. The differential system $\int_\Delta f(\lambda)\,du_m(\lambda)$ is *limited* in $(-\infty, \infty)$ if the sequence

$$\left\{ \int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)} \right\}$$

is of finite norm and norm $< M \int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[dg(\lambda)]^2}{d\rho(\lambda)}$, for every function $\Delta g(\lambda)$ integrable $H(\rho)$.
A similar definition holds for a finite interval.

THEOREM I. *The system $\Delta\rho_m(\lambda)$ is a limited differential system.*

---

* This definition is similar to that given by A. J. Pell, "Linear Equations with unsymmetric Systems of Coefficients," *Transactions of the American Mathematical Society*, Vol. 20, p. 26.

By the statements of § 1 we see immediately that $\Delta\rho_m(\lambda)$ is a differential system, and it remains to prove that it is limited. The orthogonal relation (8) is equivalent to the relation

$$(23) \qquad \int_{-\infty}^{\infty} \frac{du(\lambda)\,dv(\lambda)}{d\rho(\lambda)} = \sum_m \int_{-\infty}^{\infty} \frac{du(\lambda)\,d\rho_m(\lambda)}{d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{dv(\lambda)\,d\rho_m(\lambda)}{d\rho(\lambda)}$$

for any two functions $u(\lambda)$ and $v(\lambda)$ integrable $H(\rho)$ in $(-\infty, \infty)$.* Then, for the differential system in question,

$$\sum_m \left\{ \int_{-\infty}^{\infty} \frac{d\rho_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)} \right\}^2 = \int_{-\infty}^{\infty} \frac{[dg(\lambda)]^2}{d\rho(\lambda)}$$

for every $\Delta g(\lambda)$ integrable $H(\rho)$, therefore the condition of Definition III is satisfied, and the differential system is limited.

Definition IV. The matrix B whose coefficients are

$$(24) \qquad b_{mn} = \int_{-\infty}^{\infty} f(\lambda)\, \frac{du_m(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}$$

is the matrix corresponding to the differential system $\int_\Delta f(\lambda)\,du_m(\lambda)$.

From equation (14) of § 1 we obtain the result that *for every sequence* $\{y_m\}$ *of finite norm there exists a sequence* $\{c_a\}$ *of finite norm, and a function* $\Delta g(\lambda)$ *integrable* $H(\rho)$ *in* $(-\infty, \infty)$ *such that*

$$(25) \qquad y_m = \sum_a l_{am} c_a + \int_{-\infty}^{\infty} \frac{d\rho_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)}$$

For functions $\Delta g(\lambda)$ integrable $H(\rho)$ in $(-\infty, \infty)$ we have a similar result which we prove as the theorem

THEOREM II. *For every function* $\Delta g(\lambda)$ *integrable* $H(\rho)$ *in* $(-\infty, \infty)$ *there exists a sequence* $\{y_n\}$ *of finite norm such that*

$$\Delta g(\lambda) = \sum_n \Delta\rho_n(\lambda)\, y_n.$$

The proof follows immediately from an expansion given by Hellinger,†

$$g(\lambda) = \sum_n \rho_n(\lambda) \int_{-\infty}^{\infty} \frac{dg(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}$$

---

* Weyl, *l. c.*, *Mathematische Annalen*, Vol. 66, p. 289.

† Hellinger, *l. c.*, p. 248. The proof here given is for a finite interval, but it may be extended to include the present case.

which holds for any function $\Delta g(\lambda)$ integrable $H(\rho)$ in $(-\infty, \infty)$. The sequence

$$y_n = \int_{-\infty}^{\infty} \frac{dg(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}$$

is of finite norm by equation (23), and therefore the theorem is established.

We now show that the properties of the differential system may be derived from those of its corresponding matrix by means of the following theorem

THEOROM III. *If the differential system* $\int_{\Delta} f(\lambda)\,du_m(\lambda)$ *is limited the corresponding matrix is limited, and conversely.*

We consider the sequence $\{\sum_n b_{mn}y_n\}$ and prove that it is of finite norm for every sequence $\{y_n\}$ of finite norm. Expand the sequence $\{y_n\}$ in the form (25), then

$$\sum_n b_{mn}y_n = \sum_n \int_{-\infty}^{\infty} f(\lambda)\,\frac{du_m(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}\left[\sum_a l_{an}c_a + \int_{-\infty}^{\infty} \frac{dg(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}\right]$$

and therefore, by equations (9) and (23)

$$\sum_n b_{mn}y_n = \int_{-\infty}^{\infty} f(\lambda)\,\frac{du_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)}\,.$$

The function $\Delta g(\lambda)$ is integrable $H(\rho)$, therefore by the definition of a limited differential system the sequence $\{\sum b_{mn}y_n\}$ is of finite norm for every sequence $\{y_n\}$ of finite norm, which proves that the matrix B is limited.*

We now assume that the matrix B is limited and prove that the differential system is also limited. Let $\Delta g(\lambda)$ be any function integrable $H(\rho)$ in $(-\infty, \infty)$. By equation (23) we have

$$\int_{-\infty}^{\infty} f(\lambda)\,\frac{du_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)} = \sum_n \int_{-\infty}^{\infty} f(\lambda)\,\frac{du_m(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{dg(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}$$

and this integral exists for every function $\Delta g(\lambda)$. Hence

$$\int_{-\infty}^{\infty} f(\lambda)\,\frac{du_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)} = \sum_n b_{mn}y_n$$

---

* Hellinger and Toeplitz, "Grundlagen für eine Theorie der unendlichen Matrizen," *Mathematische Annalen*, Vol. 69, pp. 321-324.

where the sequence $\{y_n\}$ is of finite norm. The sequence $\{\sum b_{mn}y_n\}$ is of finite norm $\leqq M$, since B is limited, and it follows that the differential system is also limited.

THEOREM IV. *If the differential system* $\int_\Delta f(\lambda)du_m(\lambda)$ *is limited, the function* $\int_\Delta f(\lambda)d\sum x_m u_m(\lambda)$ *is integrable* $H(\rho)$ *in* $(-\infty, \infty)$ *for every sequence* $\{x_m\}$ *of finite norm.*

The differential system is limited and therefore the corresponding matrix B is limited. Also, for any sequence $\{x_m\}$ of finite norm, the function $\int_\Delta f(\lambda)d\sum_{m=1}^{N} x_m u_m(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$, where N has any finite value. By Theorem II there exists a sequence $\{c_p{}^N\}$ of finite norm such that

$$\int_\Delta f(\lambda)d\sum_{m=1}^{N} x_m u_m(\lambda) = \Delta \sum_p c_p{}^N \rho_p(\lambda)$$

where

$$c_p{}^N = \int_{-\infty}^{\infty} f(\lambda)\,\frac{d\sum_{m=1}^{N} x_m u_m(\lambda)\,d\rho_p(\lambda)}{d\rho(\lambda)}\,.$$

By the definition of the matrix B this gives

$$\begin{aligned}\sum_p (c_p{}^N)^2 &= \sum_p \Big(\sum_{m=1}^{N} x_m b_{mp}\Big)^2 \\ &\leq M\sum_{m=1}^{N} x_m{}^2 \\ &\leq M\sum_{m=1}^{\infty} x_m{}^2\end{aligned}$$

since B is limited, and hence the series $\sum_p c_p{}^N \rho_p(\lambda)$ converges uniformly for every finite value of N. Further

$$\lim_{N\to\infty} c_p{}^N = c_p = \sum_{m=1}^{\infty} x_m b_{mp}$$

and this sequence is of finite norm since B is limited. It follows that

$$\begin{aligned}\lim_{N\to\infty}\int_\Delta f(\lambda)\,d\sum_{m=1}^{N} x_m u_m(\lambda) &= \lim_{N\to\infty}\sum_p c_p{}^N\,\rho_p(\lambda) \\ &= \sum_p c_p\rho_p(\lambda)\end{aligned}$$

and this is sufficient to show that the function $\int_\Delta f(\lambda)\, d \sum_m x_m u_m(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$.

By the aid of this theorem we now prove the following important theorem

THEOREM V. *If* $\int_\Delta f(\lambda) du_m(\lambda)$ *is a limited differential system in* $(-\infty, \infty)$

$$\sum_m x_m \int_{-\infty}^{\infty} f(\lambda)\, \frac{du_m(\lambda) dg(\lambda)}{d\rho(\lambda)} = \int_{-\infty}^{\infty} f(\lambda)\, \frac{d \sum_m x_m u_m(\lambda) dg(\lambda)}{d\rho(\lambda)} \tag{26}$$

for every sequence $\{x_m\}$ *of finite norm, and every function* $\Delta g(\lambda)$ *integrable* $H(\rho)$ in $(-\infty, \infty)$.

Write

$$U(x, \lambda) = \sum_m x_m u_m(\lambda)$$

and consider a finite interval $(-l, l)$ of the real λ axis. Divide this interval into N parts, $\Delta_1 \cdots \Delta_N$, and let λ_i be any point of the interval Δ_i. Then for any sequence $\{x_m\}$ of finite norm, and any function $\Delta g(\lambda)$ integrable $H(\rho)$

$$\sum_{i=1}^{N} f(\lambda_i)\, \frac{\Delta_i U(x, \lambda) \Delta_i g(\lambda)}{\Delta_i \rho(\lambda)} = \sum_m x_m \sum_{i=1}^{N} f(\lambda_i)\, \frac{\Delta_i u_m(\lambda) \Delta_i g(\lambda)}{\Delta_i \rho(\lambda)} \tag{27}$$

and this is a limited linear form by Theorem IV. The coefficients

$$\sum_{i=1}^{N} f(\lambda_i)\, \frac{\Delta_i u_m(\lambda) \Delta_i g(\lambda)}{\Delta_i \rho(\lambda)}$$

form a sequence of finite norm, and

$$\lim_{N \to \infty} \sum_{i=1}^{N} f(\lambda_i)\, \frac{\Delta_i u_m(\lambda) \Delta_i g(\lambda)}{\Delta_i \rho(\lambda)} = \int_{-l}^{l} f(\lambda)\, \frac{du_m(\lambda) dg(\lambda)}{d\rho(\lambda)}$$

exists for every value of m. Then for any fixed system of values of the sequence $\{x_m\}$ we may take the limit of the right hand member of equation (27) term by term, and thus obtain the equation

$$\int_{-l}^{l} f(\lambda)\, \frac{dU(x, \lambda) dg(\lambda)}{d\rho(\lambda)} = \sum_m x_m \int_{-l}^{l} f(\lambda)\, \frac{du_m(\lambda) dg(\lambda)}{d\rho(\lambda)} \tag{28}$$

for any finite interval $(-l, l)$. Further, by Theorem IV, the integral

$$\lim_{l \to \infty} \int_{-l}^{l} f(\lambda)\, \frac{dU(x, \lambda) dg(\lambda)}{d\rho(\lambda)}$$

is of finite norm for every function $\Delta g(\lambda)$ integrable $H(\rho)$. Now if $\Delta g(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$ the function $\int_\Delta (1/\lambda)\, dg(\lambda)$ is also integrable $H(\rho)$ in $(-\infty, -\delta)$ and (δ, ∞). Hence if we write

$$\Delta h(\lambda) = \int_\Delta (1/\lambda)\, dg(\lambda)$$

the expression (30) becomes

$$\left\{ \int_{-\infty}^{-\delta} \frac{d \sum_n k_{mn}\rho_n(\lambda)\, dh(\lambda)}{d\rho(\lambda)} + \int_\delta^\infty \frac{d \sum_n k_{mn}\rho_n(\lambda)\, dh(\lambda)}{d\rho(\lambda)} \right\}$$

which is of finite norm, since we have just proved that the differential system $\Delta \sum k_{mn}\rho_n(\lambda)$ is limited, and the theorem is proved.

A similar method may be used to show that the differential system $\int_\Delta (1/|\lambda|^{\frac{1}{2}})\, d \sum_n k_{mn}\rho_n(\lambda)$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞).

4. *Expansion Theorems.* We first derive an expansion theorem for $\int_0^\pi f(x)g(x)dx$ in terms of the solutions $u_a(x)$ and $u(x, \lambda)$ of § 2, where f and g are any two functions of integrable square in $(0, \pi)$. We start from the known expansion in terms of the complete normed orthogonal system, $\phi_n(x)$, of equation (3),

$$\int_0^\pi f(x)g(x)dx = \sum_m \int_0^\pi f\phi_m \int_0^\pi g\phi_m.$$

Now the sequences $\left\{ \int_0^\pi f\phi_m \right\}$ and $\left\{ \int_0^\pi g\phi_m \right\}$ are of finite norm, therefore by equation (15)

$$\int_0^\pi fg = \sum_a \sum_m l_{am} \int_0^\pi f\phi_m \sum_n l_{an} \int_0^\pi g\phi_n + \int_{-\infty}^{\infty} \frac{d \sum_m \rho_m(\lambda) \int_0^\pi f\phi_m \; d \sum_n \rho_n(\lambda) \int_0^\pi g\phi_n}{d\rho(\lambda)}$$

We now substitute for l_{am} and $\rho_m(\lambda)$ from equations (19) and (16) respectively. This last expansion then becomes

$$\int_0^\pi f(x)g(x)dx = \sum_a \int_0^\pi f(x)u_a(x)dx \int_0^\pi g(x)u_a(x)dx + \int_{-\infty}^{\infty} \frac{d \int_0^\pi f(x)u(x,\lambda)dx \; d \int_0^\pi g(x)u(x,\lambda)dx}{d\rho(\lambda)}, \tag{31}$$

and this is the desired form.

To find a similar expansion for an arbitrary function $f(x)$, subject to suitably imposed conditions we prove some properties of the differential systems that occur in the expansion.

LEMMA I. *The differential system*

$$\int_\Delta (1/\lambda) . d \int_0^\pi (pu')' \phi_m$$

is limited in $(-\infty, \infty)$ *if we exclude an interval* $(-\delta, \delta)$ *containing the origin.*

We return to the system of equations (5), and let $\Delta = (\delta, \lambda)$ be an interval of the positive real λ axis. $[\lambda > \delta > 0]$. Then, by equation (18)

$$\int_0^\pi (pu')' \phi_m + \int_0^\pi qu\phi_m + \int_\delta^\lambda \lambda d\rho_m(\lambda) = 0. \tag{32}$$

We invert the last integral by means of a theorem of Hellinger,* namely, that if $F(\lambda) = \int_a^\lambda u(\lambda) df(\lambda)$ where $u(\lambda)$ is positive and $f(\lambda)$ is of bounded variation in (a, λ), then

$$f(\lambda) - f(a) = \int_a^\lambda \frac{dF(\lambda)}{u(\lambda)} .$$

If we write

$$-F(\lambda) = \int_0^\pi (pu')' \phi_m + \int_0^\pi qu\phi_m$$

we have

$$\rho_m(\lambda) = \int_\delta^\lambda \frac{dF(\lambda)}{\lambda} ,$$

or

$$\int_\Delta (1/\lambda)\, d \int_0^\pi (pu')' \phi_m + \int_\Delta (1/\lambda)\, d \int_0^\pi qu\phi_m + \Delta\rho_m(\lambda) = 0. \tag{33}$$

This theorem of Hellinger is also applicable if the function $u(\lambda)$ is constantly negative in the interval of integration, hence equation (33) holds for any interval Δ of the real λ axis, outside the interval $(-\delta, \delta)$.

The differential system $\Delta\rho_m(\lambda)$ is limited, by Theorem I. Also

$$\int_\Delta (1/\lambda)\, d \int_0^\pi qu\phi_m = \int_\Delta (1/\lambda)\, d \sum_n q_{mn}\rho_n(\lambda)$$

where Q is the limited matrix defined in § 1, and therefore, by Theorem VI, the differential system $\int_\Delta (1/\lambda)\, d \int_0^\pi qu\phi_m$ is limited in the intervals

* Hellinger, *l. c.*, p. 236.

$(-\infty, -\delta)$ and (δ, ∞). It follows immediately from equation (33) that the differential system $\int_\Delta (1/\lambda)\, d \int_0^\pi (pu')'\phi_m$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞).

For the characteristic solutions $u_a(x)$ belonging to the characteristic numbers λ_a we have the corresponding result that *the matrix*

$$(1/\lambda_a) \int_0^\pi (pu_a')'\phi_m$$

*is limited.**

From this result we wish to show that the differential system $\int_\Delta (m/|\lambda|^{1/2})\, d \int_0^\pi pu\phi_m$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞), but we first require the following Lemma:

LEMMA II. *The differential system*

$$\int_\Delta (1/m\,|\lambda|^{1/2})\, d \int_0^\pi (pu')'\phi_m \tag{34}$$

is limited in the intervals $(-\infty, -\delta)$ *and* (δ, ∞).

We prove this by introducing corresponding matrices [Definition IV, § 3]. Write

$$v_m(\lambda) = \int_0^\pi (pu')'\phi_m,$$

then the matrix A,

$$a_{mn} = \int_{-\infty}^{\infty} \frac{1}{\lambda}\, \frac{dv_m(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)}$$

is the limited matrix corresponding to the limited differential system $\int_\Delta (1/\lambda) dv_m(\lambda)$ of Lemma I (Theorem III). The matrix B defined by the relation

$$b_{mn} = \int_{-\infty}^{\infty} \frac{1}{m\,|\lambda|^{1/2}}\, \frac{dv_m(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)}$$

exists and is of finite norm for every value of m, and we wish to prove that it is limited. Consider the iterated matrix C

---

* The results for the characteristic solutions which we state in this section have all been proved for the non-singular equation $u'' + qu + \lambda u = 0$, by Anna Pell Wheeler, *l. c.*, *American Journal of Mathematics*, Vol. 49 (1927), pp. 317-318.

$$c_{mn} = \sum_p b_{mp} b_{np}$$

$$= \sum_p \int_{-\infty}^{\infty} \frac{1}{m\,|\,\lambda\,|^{\frac{1}{2}}} \frac{dv_m(\lambda)\,d\rho_p(\lambda)}{d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{1}{n\,|\,\lambda\,|^{\frac{1}{2}}} \frac{dv_n(\lambda)\,d\rho_p(\lambda)}{d\rho(\lambda)} .$$

By applying equation (23) we find

$$c_{mn} = \int_{-\infty}^{\infty} \frac{dv_m(\lambda)\,dv_n(\lambda)}{mn\,|\,\lambda\,|\,d\rho(\lambda)} ,$$

and by a rearrangement of terms this becomes

$$c_{mn} = \int_{-\infty}^{\infty} \frac{\dfrac{dv_m(\lambda)}{|\,\lambda\,|} \dfrac{dv_n(\lambda)}{n^2} \dfrac{n}{m}}{d\rho(\lambda)} .$$

Now consider the matrix D,

$$d_{mn} = \int_{-\infty}^{\infty} \frac{\dfrac{dv_m(\lambda)}{|\,\lambda\,|} \dfrac{dv_n(\lambda)}{n^2}}{d\rho(\lambda)}$$

$$= \sum_p \int_{-\infty}^{\infty} \frac{dv_m(\lambda)\,d\rho_p(\lambda)}{|\,\lambda\,|\,d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{dv_n(\lambda)\,d\rho_p(\lambda)}{n^2\,d\rho(\lambda)} .$$

By Lemma I the differential system $\int_\Delta (1/|\,\lambda\,|)\,dv_m(\lambda)$ is limited, and

$$v_n(\lambda)/n^2 = (1/n^2) \int_0^\pi (pu')'\phi_m$$

$$= \int_0^\pi pu \sin nx - (1/n) \int_0^\pi p'u \cos nx$$

$$= \sum_p r_{np}\rho_p(\lambda),$$

where R is the limited matrix

$$r_{np} = \int_0^\pi p(x) \sin nx \sin px - (1/n) \int_0^\pi p'(x) \cos nx \sin px.$$

Hence by Theorem VI $\Delta v_n(\lambda)/n^2$ is a limited differential system. It follows that D is the product of two limited matrices and is therefore limited. Further the matrix C is symmetric, and has been expressed in the form

$$c_{mn} = d_{mn} \cdot (n/m)$$

where D is a limited matrix. Hence C is a limited matrix,* and therefore finally B is a limited matrix, which proves the Lemma.

---

* Cf. A. J. Pell, *l. c.*, *Transactions of the American Mathematical Society*, Vol. 20, p. 35.

We now deduce the required result, namely,

LEMMA III. *The differential system*

$$(m/|\lambda|^{\frac{1}{2}})\, d \int_0^\pi pu\,\phi_m$$

is limited in the intervals $(-\infty, -\delta)$ *and* (δ, ∞).

By the preceding Lemma the differential system (34) is limited. Also

$$\int_\Delta (1/m\,|\lambda|^{\frac{1}{2}})\, d \int_0^\pi (pu')'\phi_m = \int_\Delta (m/|\lambda|^{\frac{1}{2}})\, d \int_0^\pi pu \sin mx - \int_\Delta (1/|\lambda|^{\frac{1}{2}})\, d \int_0^\pi p'u \cos mx$$

where $\int_\Delta (1/|\lambda|^{\frac{1}{2}})\, d \int_0^\pi p'u \cos mx$ is a limited differential system since it may be written in the form $\int_\Delta (1/|\lambda|^{\frac{1}{2}})\, d \sum_n t_{mn}\rho_n(\lambda)$ where T is a limited matrix (Theorem VI). We conclude immediately that the differential system $\int_\Delta (1/|\lambda|^{\frac{1}{2}})\, m\, d \int_0^\pi pu\,\phi_m$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞).

The corresponding result for the point spectrum is that *the matrix*

$$(m/|\lambda_a|^{\frac{1}{2}}) \int_0^\pi pu_a\phi_m$$

is limited.

We now determine the conditions which must be imposed on an arbitrary function $f(x)$ in order that the function $\int_\Delta |\lambda|^{\frac{1}{2}}\, d \int_0^\pi f(x)\, u(x,\lambda)$ be integrable $H(\rho)$ in $(-\infty, -\delta)$ and (δ, ∞). Return to the equation (33) and apply the theorem given by Carleman *, that if $f(\lambda)$ and $w(\lambda)$ are continuous functions in the interval (δ, λ) and $\alpha(\lambda)$ a function of bounded variation in that interval, then

$$\int_\delta^\lambda f(\lambda)\, d \int_\delta^\lambda \omega(\lambda)\, d\alpha(\lambda) = \int_\delta^\lambda f(\lambda)\omega(\lambda)\, d\alpha(\lambda).$$

For the positive interval (δ, λ) this gives

---

* Carleman, *l. c.*, p. 11.

$$-\int_\delta^\lambda (\lambda)^{\frac{1}{2}}\, d\rho_m(\lambda) = \int_\delta^\lambda (\lambda)^{\frac{1}{2}}\, d \int_\delta^\lambda (1/\lambda)\, d \int_0^\pi (pu')'\phi_m$$

$$+ \int_\delta^\lambda (\lambda)^{\frac{1}{2}}\, d \int_\delta^\lambda (1/\lambda)\, d \int_0^\pi qu\phi_m$$

$$= \int_\delta^\lambda (1/(\lambda)^{\frac{1}{2}})\, d \int_0^\pi (pu')'\phi_m + \int_\delta^\lambda (1/(\lambda)^{\frac{1}{2}})\, d \int_0^\pi qu\phi_m.$$

For the negative interval $(-\delta, -\lambda)$ we obtain similarly

$$-\int_\delta^\lambda |\lambda|^{\frac{1}{2}}\, d\rho_m(\lambda) = \int_\delta^\lambda (\lambda/|\lambda|)\ (1/|\lambda|^{\frac{1}{2}})\, d \int_0^\pi (pu')'\phi_m$$

$$+ \int_\delta^\lambda (\lambda/|\lambda|)\ (1/|\lambda|^{\frac{1}{2}})\, d \int_0^\pi qu\phi_m,$$

so that for any interval Δ outside $(-\delta, \delta)$ we have

$$\int_\Delta [1/m(|\lambda|)^{\frac{1}{2}}]\, d \int_0^\pi (pu')'\phi_m + \int_\Delta [1/m(|\lambda|)^{\frac{1}{2}}]\, d \int_0^\pi qu\phi_m$$

$$+ \int_\Delta (\lambda/m|\lambda|)\, |\lambda|^{\frac{1}{2}}\, d\rho_m(\lambda) = 0.$$

Now let $f(x)$ be a function of x such that the sequence $\{m \int_0^\pi f\phi_m\}$ is of finite norm, multiply this last equation by $m \int_0^\pi f\phi_m$ and sum for m

$$\sum_m m \int_0^\pi f\phi_m \int_\Delta [1/m(|\lambda|)^{\frac{1}{2}}]\, d \int_0^\pi (pu')'\phi_m$$

$$+ \sum_m m \int_0^\pi f\phi_m \int_\Delta [1/m(|\lambda|)^{\frac{1}{2}}]\, d \int_0^\pi qu\phi_m$$

$$+ \sum_m m \int_0^\pi f\phi_m \int_\Delta (\lambda/|\lambda|)\ (|\lambda|^{\frac{1}{2}}/m)\, d\rho_m(\lambda) = 0.$$

In each of the first two terms of this equation we have the product of a limited differential system and a sequence of finite norm, and they therefore represent functions which are integrable $H(\rho)$ in the intervals $(-\infty, -\delta)$ and (δ, ∞) (Theorem IV). It follows that *the function*

$$(\lambda/|\lambda|)\, |\lambda|^{\frac{1}{2}}\, d \int_0^\pi f(x)u(x,\lambda)dx$$

is integrable $H(\rho)$ in $(-\infty, -\delta)$ and (δ, ∞) for every function $f(x)$ such that the sequence $\{m \int_0^\pi f\phi_m\}$ is of finite norm.

For the point spectrum we have the similar result that *the sequence* $\{|\lambda_a|^{\frac{1}{2}} \int_0^\pi f(x)u_a(x)\}$ *is of finite norm for every function* $f(x)$ *such that the sequence* $\{m \int_0^\pi f\phi_m\}$ *is of finite norm.*

An expansion theorem for the function $p(x)f(x)$, where f is any function satisfying the above condition, may now be obtained, starting from the known expansion

$$p(x)f(x) = \sum_m \phi_m(x) \int_0^\pi p(x)f(x)\phi_m(x)\,dx.$$

We divide the real λ axis into three intervals $(-\infty, -\delta)$, $(-\delta, \delta)$ and (δ, ∞), and denote the characteristic values of λ belonging to these intervals by λa_1, λa_2 and λa_3, respectively. Then by the expansion (31)

$$\begin{aligned} p(x)f(x) = {} & \sum_m \phi_m(x) \sum_{a_1} \int_0^\pi p\phi_m u_{a_1} \int_0^\pi f u_{a_1} \\ & + \sum_m \phi_m(x) \int_{-\infty}^{-\delta} \frac{d \int_0^\pi p\phi_m u\, d \int_0^\pi fu}{d\rho(\lambda)} \\ & + \sum_m \phi_m(x) \sum_{a_2} \int_0^\pi p\phi_m u_{a_2} \int_0^\pi f u_{a_2} \\ & + \sum_m \phi_m(x) \int_{-\delta}^{\delta} \frac{d \int_0^\pi p\phi_m u\, d \int_0^\pi fu}{d\rho(\lambda)} \\ & + \sum_m \phi_m(x) \sum_{a_3} \int_0^\pi p\phi_m u_{a_3} \int_0^\pi f u_{a_3} \\ & + \sum_m \phi_m(x) \int_{\delta}^{\infty} \frac{d \int_0^\pi p\phi_m u\, d \int_0^\pi fu}{d\rho(\lambda)} \\ = {} & A_1 + A_2 + A_3. \end{aligned}$$

Consider first the positive interval (δ, ∞), the terms in the expansion belonging to this interval being denoted by A_3. Then

$$\begin{aligned} A_3 = {} & \sum_m \frac{\phi_m(x)}{m} \sum_{a_3} \frac{m \int_0^\pi pu_{a_3}\phi_m}{(\lambda_{a_3})^{\frac{1}{2}}} (\lambda_{a_3})^{\frac{1}{2}} \int_0^\pi f u_{a_3} \\ & + \sum_m \frac{\phi_m(x)}{m} \int_\delta^\infty \frac{\dfrac{m\, d \int_0^\pi pu\phi_m}{(\lambda)^{\frac{1}{2}}} (\lambda)^{\frac{1}{2}}\, d \int_0^\pi fu}{d\rho(\lambda)} \end{aligned}$$

and, by the properties of the differential system $\int_\Delta (1/\lambda^{1/2})\, m\, d \int_0^\pi pu\phi_m$ and of the matrix $(1/(\lambda_{a_3})^{1/2}) m \int_0^\pi pu_{a_3}\phi_m$ (Lemma III and Theorem V) we see that this may be written

$$(35) \quad A_3 = \sum_{a_3} p(x) u_{a_3}(x) \int_0^\pi f u_{a_3} + \int_\delta^\infty \frac{p(x)\, du(x,\lambda)\, d \int_0^\pi f(x) u(x,\lambda)\, dx}{d\rho(\lambda)}.$$

For the negative interval $(-\infty, -\delta)$ we find the corresponding expansion

$$(36) \quad A_1 = \sum_{a_1} p(x) u_{a_1}(x) \int_0^\pi f u_{a_1} + \int_{-\infty}^{-\delta} \frac{p(x)\, du(x,\lambda)\, d \int_0^\pi f(x) u(x,\lambda)\, dx}{d\rho(\lambda)}$$

by the same method.

We now consider the interval $(-\delta, \delta)$ in which the method of proof has to be modified since the differential system $\int_\Delta (1/|\lambda|^{1/2})\, m d \int_0^\pi pu\phi_m$ is not limited in this interval. The proof is obtained from the following Lemmas.

LEMMA IV. *The differential system* $m \int_0^\pi pu\phi_m$ *is limited in the interval* $(-\delta, \delta)$.

We return to equation (18) satisfied by $u(x,\lambda)$, and write it in the form

$$\Delta \int_0^\pi (p\phi_m')' u(x,\lambda) + \Delta \int_0^\pi q\phi_m u(x,\lambda) + \int_0^\pi \phi_m \int_\Delta \lambda du(x,\lambda) = 0.$$

In this we substitute the value of $\phi_m(x)$, and obtain the equation

$$-m\Delta \int_0^\pi pu \sin mx + \Delta \int_0^\pi p'u \cos mx + (1/m)\Delta \int_0^\pi qu \sin mx + (1/m) \int_0^\pi \sin mx \int_\Delta \lambda\, du(x,\lambda) = 0.$$

The differential systems $\Delta \int_0^\pi p'u \cos mx$ and $(1/m)\,\Delta \int_0^\pi qu \sin mx$ are limited in $(-\delta, \delta)$. Also

$$(1/m) \int_0^\pi \sin mx \int_\Delta \lambda\, du(x,\lambda) = (1/m) \int_\Delta \lambda\, d\rho_m(\lambda)$$

and this differential system is limited in $(-\delta, \delta)$. It follows immediately that the differential system $m\,\Delta \int_0^\pi pu \sin mx$ is limited in $(-\delta, \delta)$.

LEMMA V. *The function* $\Delta \int_0^\pi f(x)u(x,\lambda)\,dx$ *is integrable* $H(\rho)$ *in the interval* $(-\delta, \delta)$ *for every function* $f(x)$ *which is of integrable square in* $(0, \pi)$. For

$$\Delta \int_0^\pi f(x)u(x,\lambda) = \Delta \sum_m x_m \rho_m(\lambda),$$

where $\{x_m\}$ is a sequence of finite norm, and $\Delta\rho_m(\lambda)$ is a limited differential system. The truth of the Lemma follows immediately by Theorem IV.

For the characteristic values λ_{a_2} we have similarly that *the matrix* $m \int_0^\pi u_{a_2}\phi_m$ *is limited,* and that *the sequence* $\left[\int_0^\pi f u_{a_2}\right]$ *is of finite norm for every function* $f(x)$ *which is of integrable square in* $(0, \pi)$.

In this interval we have then

$$A_2 = \sum_m \frac{\phi_m(x)}{m} \sum_{a_2} m \int_0^\pi pu_{a_2}\phi_m \int_0^\pi f u_{a_2}$$

$$+ \sum_m \frac{\phi_m(x)}{m} \int_{-\delta}^{\delta} \frac{m d \int_0^\pi pu\phi_m\, d \int_0^\pi fu}{d\rho(\lambda)}$$

from which, by the properties we have just proved,

$$A_2 = \sum_{a_2} p(x)u_{a_2}(x) \int_0^\pi f u_{a_2} + \int_{-\delta}^{\delta} \frac{p(x)\,du(x,\lambda)\; d \int_0^\pi f(x)u(x,\lambda)\; dx}{d\rho(\lambda)} \tag{37}$$

By adding the results (35), (36) and (37) we see that the complete expansion may be expressed as the theorem.

THEOREM VII. *Let $f(x)$ be a function of x such that the sequence $\left[m \int_0^\pi f(x) \sin mx \right]$ is of finite norm, and let $u_\alpha(x)$ and $u(x, \lambda)$ be the functions defined by equations (19) and (17), respectively. Then the function $p(x)f(x)$ may be expanded in the interval $0 \leq x \leq \pi$ as the uniformly convergent series and integral*

$$(38) \qquad p(x)\, f(x) = \sum_\alpha p(x) u_\alpha(x) \int_0^\pi f u_\alpha + \int_{-\infty}^{\infty} \frac{p(x)\, du(x,\lambda)\; d \int_0^\pi f(x) u(x,\lambda)\, dx}{d\rho(\lambda)}$$

where the summation and integration extend over every part of the real λ axis in which there is a spectrum.

5. *Extension of the preceding theory to a continuous spectrum which is not simple.* We now state briefly the generalizations of the preceding theory which are required when we assume the existence of a number (at most denumerable) of continuous spectra. Let $\rho_m^{(\alpha)}(\lambda)$ be the characteristic differential forms with corresponding basis functions $\rho^{(\alpha)}(\lambda) \cdot (\alpha = 1, 2, \cdots)$. The orthogonality relations now are

$$\sum_m l_{\alpha m} l_{\beta m} = e_{\alpha\beta},$$

$$\sum_m \Delta_1 \rho_m^{(\alpha)}(\lambda)\; \Delta_2 \rho_m^{(\beta)}(\lambda) = 0 \qquad \alpha \neq \beta$$

$$= \Delta_{12} \rho^{(\alpha)}(\lambda) \qquad \alpha = \beta,$$

$$\sum_m l_{\alpha m} \rho_m^{(\beta)}(\lambda) = 0.$$

For the matrix expansions (12) and (13) we find the more general forms

$$-\sum_n (p_{mn} + q_{mn}) y_n = \sum_\alpha \lambda_\alpha Y_\alpha l_{\alpha m} + \sum_\alpha \int_{-\infty}^{\infty} \lambda \; \frac{dy^{(\alpha)}(\lambda)\; d\rho_m^{(\alpha)}(\lambda)}{d\rho^{(\alpha)}(\lambda)}$$

$$-\sum_{m,n} (p_{mn} + q_{mn}) x_m y_n = \sum_\alpha \lambda_\alpha X_\alpha Y_\alpha + \sum_\alpha \int_{-\infty}^{\infty} \lambda \; \frac{dx^{(\alpha)}(\lambda)\; dy^{(\alpha)}(\lambda)}{d\rho^{(\alpha)}(\lambda)}$$

where X and Y have the same values as before, while

$$y^{(\alpha)}(\lambda) = \sum_m \rho_m^{(\alpha)}(\lambda) y_m, \qquad x^{(\alpha)}(\lambda) = \sum_m \rho_m^{(\alpha)}(\lambda) x_m.$$

From these expansions it follows that the systems l_{am} and $\rho_m^{(\alpha)}(\lambda)$ are complete, and therefore

$$\sum_a l_{am} l_{an} + \sum_a \int_{-\infty}^{\infty} \frac{d\rho_m^{(\alpha)}(\lambda)\, d\rho_n^{(\alpha)}(\lambda)}{d\rho^{(\alpha)}(\lambda)} = e_{mn}.$$

We now have a sequence of differential solutions $u^{(\alpha)}(x, \lambda)$, defined by

$$\rho_m^{(\alpha)}(\lambda) = \int_0^{\pi} u^{(\alpha)}(x, \lambda)\, \phi_m(x)\, dx \tag{39}$$

satisfying equation (18) and the other conditions imposed on the solutions $u(x, \lambda)$ of § 2. The orthogonality relations in this case are

$$\int_0^{\pi} u_a(x) u_\beta(x)\, dx = e_{a\beta},$$

$$\int_0^{\pi} \Delta_1 u^{(\alpha)}(x, \lambda)\, \Delta_2 u^{(\beta)}(x, \lambda)\, dx = 0 \qquad \alpha \neq \beta$$

$$= \Delta_{12}\rho^{(\alpha)}(\lambda) \qquad \alpha = \beta,$$

$$\int_0^{\pi} u_a(x) \Delta u^{(\beta)}(x, \lambda) = 0.$$

The definitions of § 3 are modified as follows. We say that the sequence $\left\{ \int_\Delta f(\lambda)\, du^{(\alpha)}(\lambda) \right\}$ is *integrable* $H(\rho^{(\alpha)})$ if the integral

$$\int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[du^{(\alpha)}(\lambda)]^2}{d\rho^{(\alpha)}(\lambda)}$$

exists for every value of α, and if further the series

$$\sum_a \int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[du^{(\alpha)}(\lambda)]^2}{d\rho^{(\alpha)}(\lambda)}$$

converges.

The system of functions $\int_\Delta f(\lambda)\, du_m^{(\alpha)}(\lambda)$ is a *limited differential system* if the functions $\int_\Delta f(\lambda)\, du_m^{(\alpha)}(\lambda)$ are integrable $H(\rho^{(\alpha)})$ for every value of m, and if further the sequence

$$\left\{ \int_{-\infty}^{\infty} f(\lambda) \frac{du_m^{(\alpha)}(\lambda)\, dg^{(\alpha)}(\lambda)}{d\rho^{(\alpha)}(\lambda)} \right\}$$

is of finite norm and norm

$$\leq M \sum_{(\alpha)} \int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[dg^{(\alpha)}(\lambda)]^2}{d\rho^{(\alpha)}(\lambda)}$$

for every value of α, and every $\Delta g^{(\alpha)}(\lambda)$ integrable $H(\rho^{(\alpha)})$.

The *corresponding matrix* B is defined by

$$b_{mn} = \sum_a \int_{-\infty}^{\infty} f(\lambda) \frac{du_m^{(a)}(\lambda)\, d\rho_n^{(a)}(\lambda)}{d\rho^{(a)}(\lambda)} .$$

With these definitions the theorems of § 3 are still true, and the expansions of § 4 may be summed up as the theorem.

THEOREM VIII. *Let* $u_a(x)$ *and* $u^{(a)}(x, \lambda)$ *be the functions defined by equations* (19) *and* (39) *respectively. Then for any two functions* $f(x)$ *and* $g(x)$ *of integrable square in the interval* $(0, \pi)$

$$\int_0^\pi f(x)g(x)dx = \sum_a \int_0^\pi f(x)u_a(x)dx \int_0^\pi g(x)u_a(x)dx + \sum_a \int_{-\infty}^{\infty} \frac{d\int_0^\pi f(x)u^{(a)}(x,\lambda)dx\; d\int_0^\pi g(x)u^{(a)}(x,\lambda)dx}{d\rho^{(a)}(\lambda)} .$$

Further, if $f(x)$ *is a continuous function of* x *satisfying the boundary conditions* (2) *or* (2′) *and having a first derivative which is integrable and of integrable square in the interval* $(0, \pi)$, *then*

$$p(x)f(x) = \sum_a p(x)u_a(x) \int_0^\pi f(x)u_a(x)dx + \sum_a \int_{-\infty}^{\infty} \frac{p(x)du^{(a)}(x,\lambda)\; d\int_0^\pi f(x)u^{(a)}(x,\lambda)dx}{d\rho^{(a)}(\lambda)} ,$$

where the series and integrals involved are uniformly and absolutely convergent.

Note. Instead of equation (1) we might consider the more general equation

$$(1') \qquad (d/dx)[p(x)du/dx] + q(x)u + \lambda k(x)u = 0,$$

where k is continuous and positive in the interval $(0, \pi)$, the boundary conditions being given by equation (2) or (2′). If we assume that the function

$$\frac{1}{(k)^{\frac{1}{2}}} \frac{d}{dx}\left(p(x) \frac{d}{dx} \frac{1}{(k)^{\frac{1}{2}}} \right)$$

exists and is continuous, equation (1′) may be reduced to the form of the original equation (1)*. The solutions of equation (1′) are therefore found from the solutions of the corresponding equation (1).

* Cf. Hilbert, *l. c.*, p. 51.

In this case we find the general expansion

$$p(x)f(x) = \sum_a p(x)u_a(x)\int_0^\pi kfu_a + \sum_a \int_{-\infty}^{\infty} \frac{p(x)du^{(a)}(x,\lambda)\; d\int_0^\pi kfu^{(a)}}{d\rho^{(a)}(\lambda)}$$

where the series and integrals converge uniformly and absolutely for every function $f(x)$ satisfying the conditions previously imposed, namely, that f be continuous, satisfy the boundary conditions, and have a first derivative which is integrable and of integrable square in the interval $(0, \pi)$.

Nets of Conics in the Real Domain.

By Alan D. Campbell.

The nets of conics in the complex domain were reduced to canonical types by C. Jordan,* and for the Galois Fields $GF(p^n)$ $p > 2$ by A. H. Wilson.† The purpose of this paper is to solve the corresponding problem for the real domain. The nets are reduced to twenty-six independent types.

Denoting the net by

$$\lambda C_1 + \mu C_2 + \nu C_3 = 0 \tag{a}$$

where

$$C_i = a_i x^2 + b_i y^2 + c_i z^2 + 2f_i yz + 2g_i zx + 2h_i xy$$

$(i = 1, 2, 3)$ we apply linear transformations to x, y, z and λ, μ, ν to reduce the net to a typical form. The reduction to types is aided by considering the cubic curve $\Delta = 0$ in the plane (λ, μ, ν) whose equation is obtained by setting the discriminant of (a) equal to zero. To a line in the (λ, μ, ν) plane corresponds a pencil of the net (a) whose intersections with $\Delta = 0$ give values of λ, μ, ν for which the corresponding conic in (a) is composite. Nets of conics may therefore be divided into two main categories according as they do or do not possess a pencil of composite conics. In the former case $\Delta = 0$ is also composite. In the latter case the net may be reduced to a standard form by taking for $\lambda = 0$ the tangent at a real inflection on $\Delta = 0$, for $\mu = 0$ the harmonic polar of the point of contact of $\lambda = 0$, and for $\nu = 0$ a line through the inflection and tangent to the cubic (at a point on $\mu = 0$) or through a node or cusp (on $\mu = 0$) if the cubic have a node or cusp.

The types of pencils of conics in the real domain are known.‡ The types or pencils with $\Delta = 0$ are

(b) $\lambda x^2 + \mu y^2 = 0$ (c) $\lambda x^2 + 2\mu xy = 0$

(d) $2\lambda xy + 2\mu xz = 0$ (e) $2\lambda xy + \mu(x^2 - y^2) = 0.$

* "Réduction d'un Réseau de Formes Quadratiques," *Journal de Mathématiques*, Vol. 6, Series 2 (1906), pp. 403-438.

† "The Canonical Types of Nets of Modular Conics," *American Journal of Mathematics*, Vol. 36 (1914), pp. 187-210.

‡ L. E. Dickson, "On Families of Quadratic Forms in a General Field," *Quarterly Journal of Mathematics*, Vol. 39, pp. 316-333.

For a net containing the pencil (b) we have the general form

$$\lambda x^2 + \mu y^2 + \nu(cz^2 + 2fyz + 2gzx + 2hxy) = 0.$$

If $c \neq 0$ we can put

$$T: \quad x = x', \quad y = y', \quad z = -(g/c)x' - (f/c)y' + z';$$
$$\lambda = \lambda' + (g^2/c)\nu', \quad \mu = \mu' + (f^2/c)\nu', \quad \nu = \nu',$$

and get (after dropping the primes on the variables) the form

$$\lambda x^2 + \mu y^2 + \nu(cz^2 + 2h'xy) = 0$$

from which we derive the typical nets

(1) $\lambda x^2 + \mu y^2 + \nu z^2 = 0$
(2) $\lambda x^2 + \mu y^2 + 2\nu xy = 0$
(3) $\lambda x^2 + \mu y^2 + \nu(z^2 + 2xy) = 0.$

If $c = 0$ and f and g are not both zero, we can choose a new z and ν so as to get the form

$$\lambda x^2 + \mu y^2 + 2\nu(fyz + gzx) = 0$$

from which we derive the types

(4) $\lambda x^2 + \mu y^2 + 2\nu yz = 0$
(5) $\lambda x^2 + \mu y^2 + 2\nu(yz + zx) = 0.$

For a net containing the pencil (c) we have the general form

$$\lambda x^2 + 2\mu xy + \nu(by^2 + cz^2 + 2fyz + 2gzx) = 0.$$

If $c \neq 0$ we can subject this net to a transformation similar to the above transformation T and get the form

$$\lambda x^2 + 2\mu xy + \nu(b'y^2 + cz^2) = 0$$

and derive the new types

(6) $\lambda x^2 + 2\mu xy + \nu(z^2 - y^2) = 0$
(7) $\lambda x^2 + 2\mu xy + \nu(z^2 + y^2) = 0.$

If $c = 0$ and $f \neq 0$ in the above general form of net we can put

$$x = x', \quad y = -(g/f)x' + y', \quad z = z';$$
$$\lambda = \lambda' + 2(g/f)\mu' + b(g^2/f^2)\nu', \quad \mu = \mu' + b(g/f)\nu', \quad \nu = \nu',$$

and we get the form

$$\lambda x^2 + 2\mu xy + \nu(by^2 + 2fyz) = 0$$

and then a new z gives the type

$$\lambda x^2 + 2\mu xy + 2\nu yz = 0. \tag{8}$$

If $c = f = 0$ we have the form

$$\lambda x^2 + 2\mu xy + \nu(by^2 + 2gzx) = 0$$

and derive the new types

$$\lambda x^2 + 2\mu xy + 2\nu zx = 0 \tag{9}$$

$$\lambda x^2 + 2\mu xy + \nu(y^2 + 2xz) = 0. \tag{10}$$

For a net containing the pencil (d) we have the general form

$$2\lambda xy + 2\mu xz + \nu(ax^2 + by^2 + cz^2 + 2fyz) = 0.$$

A binary transformation on y and z and an obvious transformation on λ, μ, ν gives if $a \neq 0$ the new types

$$2\lambda xy + 2\mu xz + \nu(x^2 + y^2 + z^2) = 0 \tag{11}$$

$$2\lambda xy + 2\mu xz + \nu(x^2 - y^2 - z^2) = 0 \tag{12}$$

$$2\lambda xy + 2\mu xz + \nu(x^2 + 2yz) = 0 \tag{13}$$

$$2\lambda xy + 2\mu xz + \nu(x^2 - y^2) = 0 \tag{14}$$

and if $a = 0$

$$2\lambda xy + 2\mu xz + \nu(y^2 + z^2) = 0 \tag{15}$$

$$2\lambda xy + 2\mu xz + 2\nu yz = 0. \tag{16}$$

For a net containing the pencil (e) we have the general form

$$2\lambda xy + \mu(x^2 - y^2) + \nu(ax^2 + cz^2 + 2fyz + 2gzx) = 0.$$

If $c \neq 0$ a new z and ν give the form

$$2\lambda xy + \mu(x^2 - y^2) + \nu(a'x^2 + cz^2) = 0.$$

Hence we have the new types

$$2\lambda xy + \mu(x^2 - y^2) + \nu(z^2 + x^2) = 0 \tag{17}$$

$$2\lambda xy + \mu(x^2 - y^2) + \nu(z^2 - x^2) = 0 \tag{18}$$

$$2\lambda xy + \mu(x^2 - y^2) + \nu z^2 = 0. \tag{19}$$

If $c = 0$ we have the form

$$2\lambda xy + \mu(x^2 - y^2) + \nu(ax^2 + 2fyz + 2gzx) = 0.$$

We can put (if $g \neq 0$)*

$$x = x' - (f/g)y', \quad y = y', \quad z = z';$$
$$\lambda = \lambda', \quad \mu = (2fg/f^2 - g^2)\lambda' + \mu', \quad \nu = \nu',$$

and we get the form

$$\lambda(2\alpha x'y' + a'x'^2) + \mu(x'^2 + b'y'^2 + 2h'x'y') + \nu(a''x'^2 + b''y'^2 + 2gz'x' + 2h''x'y') = 0.$$

If we now put $x' = x''$, $y' = -(a'/\alpha)x'' + y''$, $z' = z''$, follow this by a transformation in λ, μ, ν that will rid the new C_2 and C_3 of $x''y''$ and rid C_3 of y''^2, we arrive (after an obvious transformation of the variables) at the form

$$2\lambda xy + \mu(x^2 - y^2) + \nu(a'''x^2 + 2zx) = 0$$

which reduces to (14) by choosing a new z.

When the cubic $\Delta = 0$ is non-composite it has at least one real inflectional tangent say $\lambda = 0$. The harmonic polar of the point of contact of $\lambda = 0$ may be taken to be $\mu = 0$. If one of the tangents from the point of contact, or the line joining the point of contact to a singular point, is $\nu = 0$ the equation of the cubic is now of the form

$$\Delta = \mu^2\lambda - \nu(a\lambda^2 + b\lambda\nu + c\nu^2) = 0.$$

There is no $\lambda\mu\nu$ term in Δ since every line $\lambda = \alpha\nu$ cuts the cubic and $\mu = 0$ in a harmonic set. If $\nu = 0$, then $\Delta = \lambda\mu^2$, hence the pencil $\lambda C_1 + \mu C_2 = 0$ must have $\lambda\mu^2$ as discriminant. From Dickson (*loc. cit.*, p. 318) we see that all such pencils belong to classes with typical forms reducible to

$$\lambda x^2 + 2\mu yz = 0, \quad \lambda x^2 + \mu(y^2 + z^2) = 0, \quad \text{or} \quad \lambda(x^2 \pm z^2) + 2\mu xy = 0.$$

On the other hand $\lambda = 0$ in Δ gives $\Delta = -c\nu^3$ and there are just two classes of pencils with such a discriminant, namely those with typical forms reducible to

$$\mu x^2 + \nu(y^2 + 2zx) = 0 \quad \text{or} \quad 2\mu xy + \nu(y^2 + 2zx) = 0.$$

---

* If $g = 0$, $f \neq 0$, we change λ, μ, ν so as to replace x^2 by y^2 in C_3, put $x = y'$, $y = x'$, $z = z'$, and we get the above result. If $g = f = 0$, $a \neq 0$ we have a pencil $2\lambda xy + \nu x^2 = 0$, contrary to hypothesis.

From these considerations we see that a net of conics with the above Δ can be put in one of the three following forms:

$$\lambda x^2 + 2\mu yz + \nu C_3 = 0$$
$$\lambda(x^2 - z^2) + 2\mu xy + \nu C_3 = 0$$
$$\lambda(x^2 + z^2) + 2\mu xy + \nu C_3 = 0.$$

In the first case we have the general form

$$\lambda x^2 + 2\mu yz + \nu(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0.$$

Here

$$\Delta \equiv \begin{vmatrix} a, b, c \end{vmatrix} \nu^3 + 2(gh - af)\mu\nu^2 - a\mu^2\nu + (bc - f^2)\lambda\nu^2 - \lambda\mu^2 - 2f\lambda\mu\nu = 0,$$

so that $a = f = 0$ and either g or h is zero and we have the form

$$\lambda x^2 + 2\mu yz + \nu(by^2 + cz^2 + 2hxy) = 0.$$

We therefore have the new types

(20) $$\lambda x^2 + 2\mu yz + \nu(y^2 + z^2 + 2xy) = 0$$

(21) $$\lambda x^2 + 2\mu yz + \nu(y^2 - z^2 + 2xy) = 0$$

(22) $$\lambda x^2 + 2\mu yx + \nu(z^2 + 2xy) = 0.$$

In the second case we have the general form

$$\lambda(x^2 - z^2) + 2\mu xy + \nu(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

whose discriminant is

$$\Delta \equiv \begin{vmatrix} a, b, c \end{vmatrix} \nu^3 + 2(fg - ch)\mu\nu^2 - c\mu^2\nu$$
$$+ (-ab + bc - f^2 + h^2)\lambda\nu^2 - b\lambda^2\nu + \lambda\mu^2 + 2h\lambda\mu\nu = 0.$$

Hence $h = c = 0$ and either f or g is zero. If $f = 0$ we have the form

$$\lambda(x^2 - z^2) + 2\mu xy + \nu(ax^2 + by^2 + 2gzx) = 0$$

and therefore the new type

(23) $$\lambda(x^2 - z^2) + 2\mu xy + \nu(2\alpha x^2 + y^2 + 2zx) = 0, \quad \alpha \neq 0,$$

where 2α is used instead of α to avoid fractions when solving $\mu = 0$ with $\Delta = 0$.

If $g = 0$ we have similarly the new types

(24) $$\lambda(x^2 - z^2) + 2\mu xy + \nu(\beta x^2 + y^2 + 2yz) = 0, \quad \beta \neq 0,$$

and

(25) $$\lambda(x^2 - z^2) + 2\mu xy + \nu(x^2 + 2yz) = 0,$$

according as b is not or is zero.

In the third case above we have the general form for the net

$$\lambda(x^2+z^2)+2\mu xy+\nu(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)=0$$

whose discriminant is

$$\Delta\equiv|\,a,b,c\,|\,\nu^3+2(fg-ch)\mu\nu^2-c\mu^2\nu \\ +(ab+bc-h^2-f^2)\lambda\nu^2+b\lambda^2\nu-\lambda\mu^2-2h\lambda\mu\nu=0.$$

Hence $h=c=0$ and either f or g is zero. If $f=0$ we have the form

$$\lambda(x^2+z^2)+2\mu xy+\nu(2\alpha x^2y+y^2+2zx)=0$$

with discriminant $\Delta\equiv\mu^2\lambda-\nu(\lambda^2+2\alpha\lambda\nu-\nu^2)$. The pencil $\lambda C_1+\nu C_3=0$ has three distinct degenerate conics. From Dickson we see that every such pencil belongs to one of three classes whose typical pencils are readily reducible to:

$$\lambda(x^2-y^2)+\mu(x^2-z^2)=0,\quad\text{or}\quad\lambda(x^2+y^2)+\mu(x^2-z^2)=0,$$

or $2\lambda xy+\mu(x^2-y^2+z^2)=0$, respectively. Each one of these pencils contains at least one conic that is a pair of real lines. If we take this conic for $C_1=0$ we can transform the net into (23) or (24). If $g=0$ and $b\neq 0$ we have the form

$$\lambda(x^2+z^2)+2\mu xy+\nu(\beta x^2+y^2+2yz)=0$$

with discriminant $\Delta\equiv-\lambda\mu^2+\nu(\lambda-\nu)(\lambda+\beta\nu)$. Here again the pencil $\lambda C_1+\nu C_3=0$ has one pair of real lines, and so the net can be transformed into (23) or (24). Finally if $g=b=0$ we have the new type

$$\lambda(x^2+z^2)+2\mu xy+\nu(x^2+2yz)=0. \tag{26}$$

We shall now discuss the equivalence or non-equivalence of these twenty-six typical nets. Nets with non-equivalent cubics are not equivalent, but some non-equivalent nets have the same (or equivalent) cubics. The nets (1) to (19) inclusive all have composite cubics and so are not equivalent to the nets (20) through (26). These nets with composite cubics fall into four distinct classes as follows: Class A of nets (1) to (5) inclusive where each net has a pencil (b) in it; Class B of nets (6) to (10) inclusive where each net has no pencil (b) but has a pencil (c); Class C of nets (11) to (16) inclusive where each net has no pencil (b) or (c) but has a pencil (d); Class D of nets (17), (18), (19) where each net has no pencil (b) or (c) or (d) but has a pencil (e). The cubics $\Delta=0$ of the nets in Class A serve to prove the non-equivalence of these nets to one another, except the two nets (4) and (5);

but the pencil $\lambda = 0$ in (4) cannot be transformed into the pencil $\lambda + \mu = 0$ in (5), so these two nets are non-equivalent. In Class B the nets (6) and (7) have equivalent cubics but there is no real transformation that will send (7) into (6) for any such transformation must send the pencil $\lambda x^2 + 2\mu xy = 0$ of (7) into the same pencil of (6) and so must be of the form:

$$T: \quad x = a_1 x', \quad y = a_2 x' + b_2 y', \quad z = a_3 x' + b_3 y' + c_3 z',$$

where $a_1 b_2 c_3 \neq 0$. But T cannot send the conic $z^2 + y^2 = 0$ of (7) into a conic whose y^2 and z^2 terms have opposite signs, such a conic as occurs in (6). The other nets of Class B are distinguished by their cubics. In Class C the nets all have non-equivalent cubics. In Class D the only nets with equivalent cubics are (17) and (18); but in (18) the sum of the coefficients of x^2, y^2, z^2 is identically zero in λ, μ, ν (hence we must have this sum of coefficients in any equivalent net identically zero in λ, μ, ν) also any transformation must send the pencil $2\lambda xy + \mu(x^2 - y^2) = 0$ of (17) into the same pencil of (18). There is no real non-singular transformation that will satisfy all these conditions, hence these nets are non-equivalent.

The nets (20), (21), and (22) have each a double line and so are not equivalent to the nets (23) to (26) inclusive; the cubics of these three nets with double lines have respectively a crunode, an acnode, and a cusp and so distinguish these nets. The cubic of (25) has an acnode and the cubic of (26) a crunode, so these two nets are not equivalent to each other or to the nets (23) and (24). The cubic of (23) for $\alpha = \pm 1$ has a node. If $\alpha = +1$ we can reduce (23) to (26) by the transformation

$$\lambda = -\lambda' + 2\nu', \quad \mu = -\mu', \quad \nu = \lambda' - \nu'; \quad x = -y', \quad y = x', \quad z = y' - z'$$

By a similar transformation we can reduce (23) for $\alpha = -1$ to (25). If $|\alpha| < 1$ in (23) the harmonic polar $\mu = 0$ of (0, 1, 0) cuts the cubic in one real and two conjugate imaginary points, hence for these values of α the net is not equivalent to (24). If $\beta = 1$ in (24) this net has a double line and is transformable to (21) by

$$\lambda = -\lambda', \quad \mu = \mu', \quad \nu = -\lambda' - \nu'; \quad x = z', \quad y = y', \quad z = -x' - y'.$$

If $\beta < 1$ in (24) we put

$$\lambda = \lambda', \quad \mu = [(1-\beta)^{3/4}/\beta]\mu', \quad \nu = (-1/\beta)\lambda' + [(1-\beta)^{1/2}/\beta]\nu';$$
$$x = [1/(1-\beta)^{1/4}]y', \quad y = [\beta/(1-\beta)^{1/2}]x', \quad z = -[1/(1-\beta)^{1/2}]x' + z'$$

and we get (23) for $2\alpha = (\beta - 2)/(1-\beta)^{1/2}$.

To show that for $\beta > 1$ the net (24) is not equivalent to (23) we use a projectively invariant relation connecting a line conic $Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0$ and a point conic $a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0$ and given by the equation

$$R: \quad a'A + b'B + c'C + 2f'F + 2g'G + 2h'H = 0.*$$

To a net of line conics dual to (24) for $\beta > 1$, namely

$$(24') \qquad \lambda(u^2 - w^2) + 2\mu uv + \nu(\beta u^2 + v^2 + 2vw) = 0,$$

there corresponds by R a net of point conics

$$N: \quad \lambda(-y^2 + yz) + \mu xz + \nu[(1/\beta)x^2 + (1/\beta)z^2 - y^2] = 0.$$

We can reduce N to (23) for $|\alpha| < 1$ namely $\alpha = (2-\beta)/\beta$ by the transformation

$$x = y', \quad y = x' + z', \quad z = 2x'.$$

But the cubic $\Delta = 0$ of (23) for $|\alpha| < 1$ cuts the harmonic polar $\mu = 0$ in one real and two conjugate imaginary points and therefore $\Delta = 0$ shows the non-equivalence of (23) for $|\alpha| < 1$ to (23) for $\alpha < -1$ or $\alpha > 1$. On the other hand it is easy to show that a net of line conics (23′) dual to (23) for $\alpha < -1$ or > 1 is apolar to a net of point conics (23) with $\alpha > 1$ and $\alpha < -1$ respectively. Hence (24) for $\beta > 1$ cannot be reduced to (23) for $\alpha > 1$ or $\alpha < -1$, and its cubic $\Delta = 0$ shows it is not equivalent to (23) for $|\alpha| < 1$.

* H. J. S. Smith, *Collected Works*, Vol. II, pp. 524-540.

Rational Tacnodal and Oscnodal Quartic Curves Considered as Plane Sections of Quartic Surfaces.*

By L. T. Moore and J. H. Neelley.

Introduction. It is well known that the plane sections of the Steiner Quartic Surface include all types to within projection of the rational plane quartic curve with simple singularities and that the invariants of any curve cut from this surface may be expressed as symmetric functions of the coefficients of the cutting plane.† If we vary this surface so that two of its double lines become coincident the plane sections are rational quartic curves with tacnodes; ‡ if the three double lines are made coincident the resulting surface has as plane sections rational quartic curves with oscnodes.§

It is the purpose of this paper to express the invariants of the plane sections of these two surfaces in terms of the coefficients of the cutting plane.

1. *Invariants of the Rational Tacnodal Quartic.* The equation of the tacnodal surface may be taken as

$$x_0^4 + x_1^2x_2^2 + x_0^2x_1x_3 = 0 \tag{1}$$

where $x_0x_1 = 0$ is the tacnodal line and $x_0x_2 = 0$ is the distinct double line. Two tropes of the surface are $2x_2 \pm x_3 = 0$ and the other is $x_1 = 0$ which is tangent to the surface along the tacnodal line.

The plane

$$(\alpha x) \equiv \alpha_0x_0 + \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 = 0 \tag{2}$$

cuts the surface in a quartic with a tacnode which projected upon the plane $x_3 = 0$ is

$$\alpha_3x_0^4 - \alpha_0x_0^3x_1 - \alpha_1x_0^2x_1^2 - \alpha_2x_0^2x_1x_2 + \alpha_3x_1^2x_2^2 = 0. \tag{3}$$

---

* Presented to the American Mathematical Society, May 7, 1927.

† J. E. Rowe, *Transactions of the American Mathematical Society*, Vol. 12, pp. 295-310.

‡ Salmon-Rodgers, *Analytic Geometry of Three Dimensions*, Vol. 2, p. 214; models and sections of these surfaces are given by J. H. Neelley, "Compound Singularities of the Rational Plane Quartic Curve," *American Journal of Mathematics*, Vol. 49, pp. 389-400.

§ Salmon-Rogers, *l. c.*

This quartic can also be derived from the conic

$$\alpha_1 x_0^2 - \alpha_3 x_1^2 - \alpha_3 x_2^2 + \alpha_0 x_0 x_1 + \alpha_2 x_1 x_2 = 0 \tag{4}$$

by means of the quadratic transformation *

$$x_0 = x_0' x_1', \quad x_1 = x_0'^2, \quad x_2 = x_1' x_2' \tag{5}$$

which has the inverse transformation

$$x_0' = x_0 x_1, \quad x_1' = x_0^2, \quad x_2' = x_1 x_2.$$

The transformation (5) sends (4) into a quartic with a tacnode at $(0, 0, 1)$ and the distinct node at $(0, 1, 0)$.

The invariants of (3) may be obtained from observation of the surface and the position of the cutting plane or from the conic (4) by consideration of its intersections with the reference lines. Both methods are employed in this paper.

A plane on the intersection of the nodal lines $(0, 0, 0, 1)$ cuts out a curve with a triple point and the condition is $\alpha_3 = 0$.

The three tropes meet in the point $(1, 0, 0, 0)$ and a plane on this point gives a curve which has three double lines on a point. This condition is

$$\alpha_0 = 0.$$

But a plane on either of the pinch points at the ends of the tacnodal line cuts out a ramphoid cusp and the curve with this singularity has three double lines on a point.† So the ramphoid cusp condition $\alpha_2^2 - 4\alpha_3^2 = 0$ and $\alpha_0 = 0$ form the condition for three double lines to meet in a point.

The section has an undulation when (2) touches a trope conic, that is, when

$$(\alpha_0^2 + 4\alpha_1\alpha_2 + 8\alpha_1\alpha_3)\ (\alpha_0^2 - 4\alpha_1\alpha_2 + 8\alpha_1\alpha_3)(\alpha_2^2 - 4\alpha_3^2) = 0.$$

The condition that the quartic (3) breaks up into two conics is the vanishing of the discriminant of conic (4), or

$$\tfrac{1}{4}(\alpha_1\alpha_2^2 - 4\alpha_1\alpha_3^2 - \alpha_0^2\alpha_3) = 0.$$

The conic in lines on the six flex lines of (3) is ‡

---

* Salmon-Fiedler, *Analytische Geometrie der höheren ebenen Kurven*, p. 343.

† J. H. Neelley, *l. c.*

‡ Hilton, *Plane Algebraic Curves*, p. 275.

$$(4\alpha_0^2\alpha_3^2 + 12\alpha_1\alpha_2^2\alpha_3 - 48\alpha_1\alpha_3^3)\xi_0^2 + (4\alpha_3^2 - \alpha_2^2)^2\xi_1^2$$
$$+ (4\alpha_1^2\alpha_2^2 + 12\alpha_0^2\alpha_1\alpha_3 + 48\alpha_1^2\alpha_3^2)\xi_2^2$$
$$+ 2(8\alpha_1\alpha_2\alpha_3^2 - 2\alpha_1\alpha_2^3 + 6\alpha_0^2\alpha_2\alpha_3)\xi_1\xi_2$$
$$- 32\alpha_0\alpha_1\alpha_2\alpha_3\xi_0\xi_2 + 2(16\alpha_0\alpha_3^2 - 4\alpha_0\alpha_2^2\alpha_3)\xi_0\xi_1 = 0,$$

which has the discriminant

$$144\alpha_3^2\,\Delta^2\,(4\alpha_1\alpha_2^2\alpha_3 - 16\alpha_1\alpha_3^3 - \alpha_0^2\alpha_2^2)$$

where Δ is the discriminant of (4). If degenerate curves are excluded neither α_3 nor Δ can vanish and so the condition that three flex lines meet in a point is

$$4\alpha_1\alpha_2^2\alpha_3 - 16\alpha_1\alpha_3^3 - \alpha_0^2\alpha_2^2 = 0.$$

The conic on the points of inflection is *

$$(6\alpha_0^2\alpha_2^2 + 12\alpha_0^2\alpha_3^2 - 4\alpha_1\alpha_2^2\alpha_3 + 16\alpha_1\alpha_3^3)x_0^2 + (8\alpha_1^2\alpha_2^2 - 8\alpha_0^2\alpha_1\alpha_3)x_1^2$$
$$+ (2\alpha_2^4 - 16\alpha_2^2\alpha_3^2 + 32\alpha_3^4)x_2^2$$
$$+ 2(7\alpha_0\alpha_1\alpha_2^2 + 4\alpha_0\alpha_1\alpha_3^2 - 3\alpha_0^3\alpha_3)x_0x_1$$
$$+ 2(4\alpha_0\alpha_2^3 - 16\alpha_0\alpha_2\alpha_3^2)x_0x_2$$
$$+ 2(5\alpha_1\alpha_2^3 - 20\alpha_1\alpha_2\alpha_3^2 - \alpha_0^2\alpha_2\alpha_3)x_1x_2 = 0.$$

which has the discriminant

$$36\alpha_3\,\Delta(\alpha_1^2\alpha_2^6 - 8\alpha_1^2\alpha_2^4\alpha_3^2 + 16\alpha_1^2\alpha_2^2\alpha_3^4 - 8\alpha_0^2\alpha_1\alpha_2^2\alpha_3^3$$
$$+ 32\alpha_0^2\alpha_1\alpha_3^5 - \alpha_0^4\alpha_2^2\alpha_3^2 + 8\alpha_0^4\alpha_3^4).$$

The condition that four flexes lie on a line is therefore

$$\alpha_1^2\alpha_2^6 - 8\alpha_1^2\alpha_2^4\alpha_3^2 + 16\alpha_1^2\alpha_2^2\alpha_3^4 - 8\alpha_0^2\alpha_1\alpha_2^2\alpha_3^3$$
$$+ 32\alpha_0^2\alpha_1\alpha_3^5 - \alpha_0^4\alpha_2^2\alpha_3^2 + 8\alpha_0^4\alpha_3^4 = 0.$$

The curve has a cusp if the plane is on one of the four pinch points which gives as the cusp condition

$$\alpha_1\alpha_3(\alpha_2^2 - 4\alpha_3^2) = 0.$$

The curve has two cusps if

$$\alpha_1 = 0 \quad \text{and} \quad \alpha_2^2 - 4\alpha_3^2 = 0.$$

These invariants are sufficient to set up a correspondence between them and the invariants of the fundamental involution of the rational plane quartic curve. Since Δ does not vanish for a proper quartic it is used to raise the invariants derived from the surface to the proper degree.

---

* Richmond and Stuart, *Proceedings of the London Mathematical Society*, Vol. 2, pp. 131-133; Hilton, *l. c.*, pp. 275-276.

This correspondence is given in the table below:

Singularity	*Invts. of Fund. Involution*	*Invts. from Surface*
Triple Point	I_2	$\alpha_3(\alpha_1\alpha_2^2-4\alpha_1\alpha_3^2-\alpha_0^2\alpha_3)$.
Disc. of q	I_2'	$3\alpha_0^2\alpha_3^2+\alpha_1\alpha_2^2\alpha_3-4\alpha_1\alpha_3^3-\alpha_0^2\alpha_2^2$
Disc. of $N=0$	$I_2'-4I_2$	$4\alpha_0^2\alpha_3^2-\alpha_0^2\alpha_2^2$
Disc. of $g_2=0$	$I_2'+12I_2$	$4\alpha_1\alpha_2^2\alpha_3-16\alpha_1\alpha_3^3-\alpha_0^2\alpha_2^2$
Undulation	I_4	$(\alpha_0^2+4\alpha_1\alpha_2+8\alpha_1\alpha_3)(\alpha_0^2-4\alpha_1\alpha_2+8\alpha_1\alpha_3)(\alpha_2^2-4\alpha_3^2)$
Disc. of $F=0$	$I_4-(I_2'-36I_2)^2$	$(\alpha_1^2\alpha_2^6-8\alpha_1^2\alpha_2^4\alpha_3^2+16\alpha_1^2\alpha_2^2\alpha_3^4-8\alpha_0^2\alpha_1\alpha_2^2\alpha_3^3$ $+32\alpha_0^2\alpha_1\alpha_3^5-\alpha_0^4\alpha_2^2\alpha_3^2+8\alpha_0^4\alpha_3^4)$
Cusp	I_6	$\alpha_1\alpha_3(\alpha_2^2-4\alpha_3^2)^2(\alpha_1\alpha_2^2-4\alpha_1\alpha_3^2-\alpha_0^2\alpha_3)^2$

The two sets of invariants are connected by the relations

$$
\begin{aligned}
4I_2 &= M(\alpha_1\alpha_2^2\alpha_3 - 4\alpha_1\alpha_3^3 - \alpha_0^2\alpha_3^2).\\
I_2' &= M(3\alpha_0^2\alpha_3^2 + \alpha_1\alpha_2^2\alpha_3 - 4\alpha_1\alpha_3^3 - \alpha_0^2\alpha_2^2).\\
I_4 &= M^2(\alpha_0^2 + 4\alpha_1\alpha_2 + 8\alpha_1\alpha_3)(\alpha_0^2 - 4\alpha_1\alpha_2 + 8\alpha_1\alpha_3)(\alpha_2^2 - 4\alpha_3^2)^2.\\
16I_6 &= -M^3\alpha_1\alpha_3(\alpha_2^2 - 4\alpha_3^2)^2\ (\alpha_1\alpha_2^2 - 4\alpha_1\alpha_3^2 - \alpha_0^2\alpha_3)^2.
\end{aligned}
\tag{6}
$$

where M is a positive constant and the other constant factors are such that the tacnode condition $I_6'=0$ is satisfied.

2. *An Invariant Classification of Tacnodal Quartics.* There are two types of tacnodes, the embrassement and the opposition. At an embrassement both branches of the curve lie on the same side of the tangent at the tacnode; at an opposition the branches lie on opposite sides of the tangent. Each proper quartic cut from surface (1) has an embrassement.

However, every section of the surface

$$x_0^4 + x_0^2x_1x_3 - x_1^2x_2^2 = 0 \tag{7}$$

by the plane $(\beta x)=0$ is a quartic with an opposition.* This surface may be derived from (1) by the transformation

$$x_0 = ix_0', \quad x_1 = ix_1', \quad x_2 = x_2', \quad x_3 = ix_3'$$

which gives

$$\alpha_0 = i\beta_0, \quad \alpha_1 = i\beta_1, \quad \alpha_2 = \beta_2, \quad \alpha_3 = i\beta_3 \tag{8}$$

---

* R. Gentry, *A Dissertation on the Forms of Plane Quartic Curves*, (Bryn Mawr, 1896), pp. 31-35.

From equations (6) it can be shown that

$$M\,\alpha_0^2\alpha_2^2 = \frac{(I_2'+12I_2)(I_2'-4I_2)I_2^2-I_6(I_2'-4I_2)}{I_6-(I_2'-4I_2)I_2^2},$$

and

$$M\,\alpha_0^2\alpha_3^2 = \frac{(I_2'+12I_2)(I_2'-4I_2)I_2^2-I_2^2(I_2'-4I_2)^2}{4\{I_6-(I_2'-4I_2)I_2^2\}},$$

Hence

$$\alpha_2^2/\alpha_3^2 = \frac{(I_2'+12I_2)I_2^2-I_6}{4I_2^3}.$$

This invariant by virtue of the relations (8) is greater than zero for the quartic with an embrassement and less than zero for the quartic with an opposition.

3. *Invariants of the Rational Oscnodal Quartic.* The oscnodal quartic surface has the equation *

$$x_0^3x_3 - (x_1^2 - x_0x_2)^2 = 0. \tag{9}$$

The projection upon the plane $x_3 = 0$ of its section by the plane $(\alpha x) = 0$ gives

$$\alpha_0x_0^4 + \alpha_1x_0^3x_1 + \alpha_2x_0^3x_2 + \alpha_3x_0^2x_2^2 - 2\alpha_3x_0x_1^2x_2 + \alpha_3x_1^4 = 0. \tag{10}$$

This quartic may also be derived from the conic

$$\alpha_2x_0^2 + \alpha_0x_1^2 + \alpha_3x_2^2 + \alpha_1x_0x_1 + \alpha_2x_1x_2 = 0 \tag{11}$$

by the transformation †

$$x_0 = x_0'x_1', \quad x_1 = x_0'^2, \quad x_2 = x_0'x_2' - x_1'^2 \tag{12}$$

whence

$$x_0' = x_1^2, \quad x_1' = x_0x_1, \quad x_2' = x_1x_2 + x_0^2.$$

The surface (9) has three coincident double lines which form the oscnodal line

$$x_0x_1 = 0$$

and one trope

$$x_3 = 0.$$

The quartic (10) accordingly has an oscnode at $(0, 0, 1)$.

The methods of the first section of this paper give the correspondence of the two sets of invariants as tabulated below.

---

* Salmon-Rogers, *l. c.*

† Salmon-Fiedler, *l. c.*

Invariants of Fund. Involution	*Invariants from Surface* (9)
I_2	$\alpha_1^2\alpha_3^2 + \alpha_2^3\alpha_3 - 4\alpha_0\alpha_2\alpha_3^2$
I_2'	$\alpha_1^2\alpha_3^2 - 15\alpha_2^3\alpha_3 - 4\alpha_0\alpha_2\alpha_3^2$
$I_2' - 4I_2$	$\alpha_2^3\alpha_3$
$I_2' + 12I_2$	$\alpha_1^2\alpha_3^2 - 4\alpha_0\alpha_2\alpha_3^2 - 3\alpha_2^3\alpha_3$
I_4	$\alpha_2^3\alpha_3^3(4\alpha_0\alpha_2 - \alpha_1^2)$
	$40\alpha_0\alpha_2^4\alpha_3^3 - 10\alpha_1^2\alpha_2^3\alpha_3^3 - \alpha_1^4\alpha_3^4 - 9\alpha_2^6\alpha_3^2$
$I_4 - (I_2' - 36I_2)^2$	$- 16\alpha_0^2\alpha_2^2\alpha_3^4 + 8\alpha_0\alpha_1^2\alpha_2\alpha_3^4$
I_6	$\alpha_2^3\alpha_3^3(4\alpha_0\alpha_2\alpha_3 - \alpha_1^2\alpha_3 - \alpha_2^3)^2$.

The relations connecting the invariants are

$$
\begin{aligned}
16I_2 &= M(\alpha_1^2\alpha_3^2 + \alpha_2^3\alpha_3 - 4\alpha_0\alpha_2\alpha_3^2) \\
4I_2' &= M(\alpha_1^2\alpha_3^2 - 15\alpha_2^3\alpha_3 - 4\alpha_0\alpha_2\alpha_3^2) \\
I_4 &= 16M^2\,\alpha_2^3\alpha_3^3(4\alpha_0\alpha_2 - \alpha_1^2) \\
64I_6 &= -M^3\alpha_2^3\alpha_3^3\,(4\alpha_0\alpha_2\alpha_3 - \alpha_1^2\alpha_3 - \alpha_2^3)^2
\end{aligned}
\tag{13}
$$

where the constants have been chosen so that the invariants satisfy the oscnodal conditions *

$$
\begin{cases}
I_2I_4 - I_2(I_2' - 4I_2)^2 - 64I_6 = 0. \\
(I_2' - 4I_2)\,(I_2' + 60I_2) - I_4 = 0.
\end{cases}
$$

M is a negative constant here.

YALE UNIVERSITY.
CARNEGIE INSTITUTE OF TECHNOLOGY.

* J. H. Neelley, *l. c.*